Robust Stability for Delay Lur'e Control Systems with Multiple Nonlinearities

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Abstract

For delay Lur'e control systems with multiple non-linearities and time-varying uncertainties, necessary and sufficient conditions for the existence of Lyapunov functional in the extended Lur'e form with negative definite derivative to guarantee robust absolute stability are derived by solving a set of linear matrix inequalities(LMIs). Moreover, some new delay-dependent absolute stability criteria are also presented, in which some free weighting matrices that express the relationships between the terms in Leibniz-Newton formula are considered. Finally, an example is provided to illustrate the effectiveness of the proposed method.

Key words: Lur'e control systems, absolute stability, robustness, Lyapunov functional, linear matrix inequality, delay-dependent criterion

1 Introduction

The absolute stability problem of Lur'e control systems has widely been studied and has practical applications(see [15], [13], [8], [20], [7]). Since time-delay is commonly encountered in various engineering systems and it is frequently a source of instability and so the stability problem of delay Lur'e control systems has been of interest to researchers over the past decades (see [16], [1], [6], [2], [9]). Some necessary and sufficient conditions are given for the existence of Lyapunov functional in the extended Lur'e form for delay Lur'e control systems with negative definite derivative to guarantee absolute stability (see [6]). These results provide only the existence conditions, and are not solvable.

On the other hand, the necessary and sufficient conditions in [6] can not be extended to the systems with time-varying structured uncertainties. [9] employed Linear matrix inequality(LMI) to express the necessary and sufficient conditions obtained in [6]. The advantage of this work is that the free parameters in the Lyapunov functional can be derived by solving a group of LMIs. By the way, the results in [9] can be easily extended to the systems with time-varying structured uncertainties, which will be given this paper.

Moreover, the criteria which do not include the information on delay are called delay-independent criteria. They are more conservative than the delay-dependent criteria when delays guaranteeing stability are small. It is important to discuss the delay-dependent problem of delay Lur'e control systems. Recently, there are a number of interesting new ideas to improve the results on delay-dependent stability of linear systems with delay (see [17], [3], [14], [12], [11], [19], [4], [5]). The most effective methods are presented by [14] and extended by [12]. In the derivative of Lyapunov functional, they used the Leibniz-Newton formula and replaced the term $x(t-\tau)$ with $x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$ in some places, but kept it in other places. For example, in [12], $x(t-\tau)$ in the expression $2x^T(t)PA_1\dot{x}(t)$ is replaced with $x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$; but in $\tau\dot{x}^T(t)Z\dot{x}(t)$, it is not. In fact, there must exist some optimal weighting matrices between the terms in the Leibniz-Newton formula, but they gave some fixed weighting matrices. Recently, [10] presented a new method that use the free weighting matrices to express the relationship between the terms in the Leibniz-Newton formula. In addition, the optimal weighting matrices can be solved by the solutions of some LMIs.

This paper discusses the existence problem of a Lyapunov functional in the extended Lur'e form with negative definite derivative to guarantee robust absolute stability of delay Lur'e control systems with multiple non-linearities in the bounded sector. Some necessary and sufficient conditions for the existence problem are presented. They convert the problem to one of solving a set of LMIs. In addition, the method presented in [10], in which some free weighting matrices are used to express the relationships between the terms in Leibniz-Newton formula, is employed to derive the delay-dependent robust absolute stability criteria. Finally, an example is proposed to illustrate the improvement of the necessary and sufficient condition over the sufficient condition directly using S-procedure. The benefit of delay-dependent criteria is also demonstrate in the example.

2 Notation and Preliminaries

Consider a time-varying structured uncertain delay Lur'e control systems with multiple non-linearities

$$\mathcal{S}_1 : \begin{cases} \dot{x}(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t-\tau) \\ \qquad\quad +(D + \Delta D(t))f(\sigma(t)), \\ \sigma(t) = C^T x(t), \end{cases} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is the state vector, $\tau > 0$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $D = (d_{ij})_{n\times m} = (d_1, d_2, \cdots, d_m)$, $C = (c_{ij})_{n\times m} = (c_1, c_2, \cdots, c_m)$, d_j and $c_j (j = 1, 2, \cdots, m)$ are the j-th column of D and C, respectively, $\sigma(t) = (\sigma_1(t), \sigma_2(t), \cdots, \sigma_m(t))^T$, $f(\sigma(t)) = (f_1(\sigma_1(t)), f_2(\sigma_2(t)), \cdots, f_m(\sigma_m(t)))^T$ is the nonlinear function.

The nominal form of system $\mathcal{S}_1$ is given by

$$\mathcal{S}_0 : \begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau) + Df(\sigma(t)), \\ \sigma(t) = C^T x(t). \end{cases} \tag{2}$$

Here, the non-linearities $f_j(\cdot)$ satisfy the following

$$f_j(\cdot) \in K_j[0, k_j] = \left\{ f_j(\sigma_j) | f_j(0) = 0, 0 \le \sigma_j f_j(\sigma_j) \le k_j \sigma_j^2 (\sigma_j \ne 0) \right\}, \quad j = 1, 2, \cdots, m, \tag{3}$$

with $0 < k_j < +\infty, j = 1, 2, \cdots, m$. For simplicity, sometimes we denote $f_j(\sigma_j) = f_j(\sigma_j(t))$.

Also, the uncertainties are assumed to be of the following form

$$[\Delta A(t)\ \Delta B(t)\ \Delta D(t)] = HF(t)[E_a\ E_b\ E_d], \tag{4}$$

where H, E_a, E_b, E_d are the known real constant matrices with appropriate dimensions, E_{dj} is the j-th column of E_d, $F(t)$ is an unknown real time-varying matrix with Lebesgue measurable elements satisfying

$$\|F(t)\| \le 1, \quad \forall\ t. \tag{5}$$

For the sake of simplicity, let

$$\bar{A} = A + \Delta A(t),\ \ \bar{B} = B + \Delta B(t),\ \ \bar{D} = D + \Delta D(t). \tag{6}$$

Construct the Lyapunov functional in the extended Lur'e form as

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Qx(s) + 2\sum_{j=1}^{m} \lambda_j \int_{0}^{\sigma_j} f_j(\sigma_j)d\sigma_j, \tag{7}$$

where $P = P^T > 0$, $Q = Q^T > 0$ and $\lambda_j \geq 0 (j = 1, 2, \cdots, m)$ need to be determined.

Definition 1 . *The functional $V(x_t)$ of (7) is said to be a Lyapunov functional of system $\mathcal{S}_1$ (or nominal system $\mathcal{S}_0$) with negative definite derivative, that is,*

$$\begin{aligned} &\dot{V}(x_t)|_{\mathcal{S}_1} < 0\ (or \dot{V}(x_t)|_{\mathcal{S}_0} < 0), \quad on\ K = diag(k_1, k_2, \cdots, k_m), \\ &\quad if\ for\ any\ f_j(\cdot) \in K_j[0, k_j](j = 1, 2, \cdots, m), (x(t), x(t-\tau)) \neq 0. \end{aligned} \tag{8}$$

If condition (8) holds, system $\mathcal{S}_1$(or nominal system $\mathcal{S}_0$) is robustly absolutely stable(or absolutely stable) in the sector bounded by $K = \mathrm{diag}(k_1, k_2, \cdots, k_m)$.

To derive the main results in the next section, the following lemmas are given.

Lemma 2 ([9]): *Equation (8) for nominal system $\mathcal{S}_0$ holds, which ensure system $\mathcal{S}_0$ is absolutely stable in the sector bounded by K =diag$(k_1, k_2, \cdots, k_m)$, if there exist $P = P^T > 0$,$Q = Q^T > 0$, T=diag$(t_1, t_2, \cdots, t_m) \geq 0$, Λ=diag $(\lambda_1, \lambda_2, \cdots, \lambda_m) \geq 0$ such that the LMI*

$$\Omega = \begin{bmatrix} A^TP + PA + Q & PB & PD + A^TC\Lambda + CKT \\ B^TP & -Q & B^TC\Lambda \\ D^TP + \Lambda C^TA + TKC^T & \Lambda C^TB & \Lambda C^TD + D^TC\Lambda - 2T \end{bmatrix} < 0, \tag{9}$$

holds. It is necessary while $m = 1$.

Let α =diag$(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and

$$\begin{aligned} D_j^m = \{\alpha | \alpha_i = 0, for\ i \geq j;\ \ \alpha_i \in \{0, k_i\}, for\ i < j, (i = 1, 2, \cdots, m)\}, \\ j = 1, 2, \cdots, m. \end{aligned} \tag{10}$$

with 2^{j-1} elements. Assume that

$$A(\alpha) := A + D\alpha C^T, \quad P(\alpha) := P + C\Lambda\alpha C^T. \tag{11}$$

Then, we have the following Lemma

Lemma 3 ([9]): *It assumes that $m \geq 1$ for nominal system $\mathcal{S}_0$. The necessary and sufficient conditions for the existence of Lyapunov functional $V(x_t)$ in (7) satisfying equation (8), which ensure system $\mathcal{S}_0$ is absolutely stable in the sector bounded by $K =$diag$(k_1, k_2, \cdots, k_m)$, are that $\forall \alpha \in D_j^m (j = 1, 2, \cdots, m)$, there exist $t_\alpha \geq 0$ and $P = P^T > 0$ and $Q = Q^T > 0$ and $\lambda_i \geq 0 (i = 1, 2, \cdots, m)$, such that LMIs in the following hold.*

$$G_j(\alpha) = \begin{bmatrix} \Phi_{11} & P(\alpha)B & \Phi_{13} + t_\alpha k_j c_j \\ B^T P(\alpha) & -Q & \lambda_j B^T c_j \\ \Phi_{13}^T + t_\alpha k_j c_j^T & \lambda_j c_j^T B & 2\lambda_j c_j^T d_j - 2t_\alpha \end{bmatrix} < 0, \tag{12}$$

where

$$\Phi_{11} = A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) + Q,$$
$$\Phi_{13} = P(\alpha)d_j + \lambda_j A^T(\alpha)c_j.$$

To obtain the conditions for the systems with time-varying structured uncertainties, the following lemma is needed to deal with the uncertainties.

Lemma 4 ([18]): *Given matrices $Q = Q^T, H, E$ and $R = R^T > 0$ of appropriate dimensions, then*

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying $F^T F \leq R$, if and only if there exists some $\varepsilon > 0$ such that

$$Q + \varepsilon H H^T + \varepsilon^{-1} E^T R E < 0.$$

3 Robustness for Absolute Stability

For a time-varying structured uncertain system $\mathcal{S}_1$, the following sufficient condition is derived from Lemma 2 which uses S-procedure directly for the non-linearities, in which the uncertainties are dealt with by using Lemma 4.

Theorem 5 *System $\mathcal{S}_1$ is robustly absolutely stable in the sector bounded by K =diag$(k_1, k_2, \cdots, k_m)$ if there exist $P = P^T > 0$, $Q = Q^T > 0$, Λ=diag $(\lambda_1, \lambda_2, \cdots, \lambda_m) \geq 0$,$T$=diag$(t_1, t_2, \cdots, t_m) \geq 0$ and $\varepsilon \geq 0$ such that the LMI*

$$\begin{bmatrix} \Psi_{11} & PB + \varepsilon E_a^T E_b & \Psi_{13} & PH \\ B^T P + \varepsilon E_b^T E_a & -Q + \varepsilon E_b^T E_b & B^T C\Lambda + \varepsilon E_b^T E_d & 0 \\ \Psi_{13}^T & \Lambda C^T B + \varepsilon E_d^T E_b & \Psi_{33} & \Lambda C^T H \\ H^T P & 0 & H^T C\Lambda & -\varepsilon I \end{bmatrix} < 0 \qquad (13)$$

holds, where

$$\begin{aligned} \Psi_{11} &= A^T P + PA + Q + \varepsilon E_a^T E_a, \\ \Psi_{13} &= PD + A^T C\Lambda + CKT + \varepsilon E_a^T E_d, \\ \Psi_{33} &= \Lambda C^T D + D^T C\Lambda - 2T + \varepsilon E_d^T E_d. \end{aligned}$$

Proof: Replacing A, B, D in (9) with $A + HF(t)E_a$, $B + HF(t)E_b$ and $D + HF(t)E_d$, respectively, we find that (9) for $\mathcal{S}_1$ is equivalent to the following condition

$$\Omega + \begin{bmatrix} PH \\ 0 \\ \Lambda C^T H \end{bmatrix} F(t) \begin{bmatrix} E_a & E_b & E_d \end{bmatrix} + \begin{bmatrix} E_a^T \\ E_b^T \\ E_d^T \end{bmatrix} F^T(t) \begin{bmatrix} H^T P & 0 & H^T C\Lambda \end{bmatrix} < 0. \qquad (14)$$

By Lemma 4, a necessary and sufficient condition guaranteeing (14) is that there exists a positive number $\varepsilon > 0$ such that

$$\Omega + \varepsilon^{-1} \begin{bmatrix} PH \\ 0 \\ \Lambda C^T H \end{bmatrix} \begin{bmatrix} H^T P & 0 & H^T C\Lambda \end{bmatrix} + \varepsilon \begin{bmatrix} E_a^T \\ E_b^T \\ E_d^T \end{bmatrix} \begin{bmatrix} E_a & E_b & E_d \end{bmatrix} < 0. \qquad (15)$$

Applying Schur complements shows that (15) is equivalent to (13). $\square$

The above theorem is conservative to examine robust absolute stability of system $\mathcal{S}_1$ with multiple non-linearities since it is only based on some sufficient conditions. Now, the necessary and sufficient condition for system $\mathcal{S}_1$ is derived, based on the necessary and sufficient condition in Lemma 3.

Theorem 6 *The necessary and sufficient conditions for the existence of Lyapunov functional $V(x_t)$ in (7) satisfying equation (8), which ensure system $\mathcal{S}_1$*

is robustly absolutely stable in the sector bounded by $K = \text{diag}(k_1, k_2, \cdots, k_m)$ *are that* $\forall \alpha \in D_j^m (j = 1, 2, \cdots, m)$, *there exist* $t_\alpha \geq 0$ *and* $P = P^T > 0$ *and* $Q = Q^T > 0$ *and* $\lambda_i \geq 0 (i = 1, 2, \cdots, m)$ *and* $\varepsilon_\alpha \geq 0$, *such that LMIs in the following hold.*

$$\hat{G}_j(\alpha) = \begin{bmatrix} \hat{\Phi}_{11} & \hat{\Phi}_{12} & \hat{\Phi}_{13} & P(\alpha)H \\ \hat{\Phi}_{12}^T & \hat{\Phi}_{22} & \hat{\Phi}_{23} & 0 \\ \hat{\Phi}_{13}^T & \hat{\Phi}_{23}^T & \hat{\Phi}_{33} & \lambda_j c_j^T H \\ H^T P(\alpha) & 0 & \lambda_j H^T c_j & -\varepsilon_\alpha I \end{bmatrix} < 0, \tag{16}$$

where

$$\begin{aligned} &\hat{\Phi}_{11} = \Phi_{11} + \varepsilon_\alpha E_a^T(\alpha) E_a(\alpha), \\ &\hat{\Phi}_{12} = P(\alpha)B + \varepsilon_\alpha E_a^T(\alpha) E_b, \\ &\hat{\Phi}_{13} = \Phi_{13} + t_\alpha k_j c_j + \varepsilon_\alpha E_a^T(\alpha) E_{dj}, \\ &\hat{\Phi}_{22} = -Q + \varepsilon_\alpha E_b^T E_b, \\ &\hat{\Phi}_{23} = \lambda_j B^T c_j + \varepsilon_\alpha E_b^T E_{dj} \\ &\hat{\Phi}_{33} = 2\lambda_j c_j^T d_j - 2t_\alpha + \varepsilon_\alpha E_{dj}^T E_{dj}, \\ &E_a(\alpha) = (E_a + E_d \alpha C^T)^T (E_a + E_d \alpha C^T), \end{aligned}$$

and Φ_{11} *and* Φ_{13} *are defined in (12).*

Proof: Let $\bar{A}(\alpha) = \bar{A} + \bar{D}\alpha C^T$ and $\bar{d}_j$ is the j-th column of $\bar{D}$. From Lemma 3, we find the conditions (12) for $\mathcal{S}_1$ are equivalent to that there exist $P = P^T > 0$, $Q = Q^T > 0$, $\lambda_i \geq 0 (i = 1, 2, \cdots, m)$, $\forall j = 1, 2, \cdots, m$ and $\forall \alpha \in D_j^m$ there exist t_α such that the following holds.

$$\bar{G}_j(\alpha) = \begin{bmatrix} \bar{\Phi}_{11} & P(\alpha)\bar{B} & \bar{\Phi}_{13} + t_\alpha k_j c_j \\ \bar{B}^T P(\alpha) & -Q & \lambda_j \bar{B}^T c_j \\ \bar{\Phi}_{13}^T + t_\alpha k_j c_j^T & \lambda_j c_j^T \bar{B} & 2\lambda_j c_j^T \bar{d}_j - 2t_\alpha \end{bmatrix} < 0, \tag{17}$$

where

$$\begin{aligned} &\bar{\Phi}_{11} = \bar{A}^T(\alpha) P(\alpha) + P(\alpha)\bar{A}(\alpha) + Q, \\ &\bar{\Phi}_{13} = P(\alpha)\bar{d}_j + \lambda_j \bar{A}^T(\alpha) c_j. \end{aligned}$$

Replacing $\bar{A}(\alpha)$, $\bar{B}$ and $\bar{d}_j$ in (17) with $A(\alpha)+HF(t)E_a(\alpha)$, $B+HF(t)E_b$ and $d_j + HF(t)E_{dj}$, respectively, $\bar{G}_j(\alpha)$ can be rewritten as

$$\begin{aligned}\bar{G}_j(\alpha) = G_j(\alpha) + &\begin{bmatrix} P(\alpha)H \\ 0 \\ \lambda_j c_j^T H \end{bmatrix} F(t) \begin{bmatrix} E_a(\alpha) & E_b & E_{dj} \end{bmatrix} \\ &+ \begin{bmatrix} E_a^T(\alpha) \\ E_b^T \\ E_{dj}^T \end{bmatrix} F^T(t) \begin{bmatrix} H^T P(\alpha) & 0 & \lambda_j H^T c_j \end{bmatrix}.\end{aligned} \tag{18}$$

where $G_j(\alpha)$ is defined in (12). Then, by Lemma 4 and Schur complements, $\bar{G}_j(\alpha) < 0$ if and only if LMI (16) are true. □

Remark 1: Theorem 6 is based on the necessary and sufficient conditions for nominal system $\mathcal{S}_0$. The improvement over the Theorem 5 which is a sufficient condition will be shown in the example.

4 Delay Dependent Conditions

The criteria given in the previous section do not include information on delay, which are referred to as delay-independent criteria. Sometimes, the system $\mathcal{S}_1$ or $\mathcal{S}_0$ are absolutely stable while $\tau = 0$, but they are not absolutely stable for all $\tau > 0$, it follows from the continues that the systems are absolutely stable when τ is very small. So, the criteria that don't include information on delay are more conservative than that include information on delay which are referred to as delay-dependent criteria. Many authors presented some delay-dependent criteria for the linear system. Followed by the method presented in [10], the following theorem also attempts to take this relationship between the terms in Leibniz-Newton formula into account for delay Lur'e control systems. Specifically, the expression $2\left[x^T(t)N_1 + x^T(t-\tau)N_2 + f^T(\sigma(t))N_3\right]\left[x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds - x(t-\tau)\right]$, which is equal to zero, is added to $\dot{V}(x_t)$, and a new delay-dependent absolute stability criterion is derived.

Theorem 7 *Given a scalar $\tau > 0$, system $\mathcal{S}_0$ is absolutely stable if there exist*

$P = P^T > 0,\ Q = Q^T > 0,\ Z = Z^T \geq 0,\ X = X^T = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0,$
Λ=diag $(\lambda_1, \lambda_2, \cdots, \lambda_m) \geq 0$, T=diag$(t_1, t_2, \cdots, t_m) \geq 0$ and any matrices $N_i (i = 1, 2, 3)$ such that the following LMIs (19) and (20) hold.

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} + CKT & \tau A^T Z \\ \Gamma_{12}^T & \Gamma_{22} & \Gamma_{23} & \tau B^T Z \\ \Gamma_{13}^T + TKC^T & \Gamma_{23}^T & \Gamma_{33} - 2T & \tau D^T Z \\ \tau ZA & \tau ZB & \tau ZD & -\tau Z \end{bmatrix} < 0 \tag{19}$$

$$\Pi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ X_{12}^T & X_{22} & X_{23} & N_2 \\ X_{13}^T & X_{23}^T & X_{33} & N_3 \\ N_1^T & N_2^T & N_3^T & Z \end{bmatrix} \geq 0, \tag{20}$$

where

$$\begin{aligned} \Gamma_{11} &= A^T P + PA + Q + N_1 + N_1^T + \tau X_{11}, \\ \Gamma_{12} &= PB + N_2^T - N_1 + \tau X_{12}, \\ \Gamma_{13} &= PD + A^T C\Lambda + N_3^T + \tau X_{13}, \\ \Gamma_{22} &= -Q - N_2 - N_2^T + \tau X_{22}, \\ \Gamma_{23} &= B^T C\Lambda - N_3^T + \tau X_{23}, \\ \Gamma_{33} &= \Lambda C^T D + D^T C\Lambda + \tau X_{33}. \end{aligned}$$

Proof: Construct Lyapunov functional candidate as

$$V_d(x_t) = V(x_t) + \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Z \dot{x}(s) ds d\theta, \tag{21}$$

where $V(x_t)$ is defined in (7) and $Z = Z^T \geq 0$ need to be determined.

Using the Leibniz-Newton formula one can write

$$x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(s) ds = 0. \tag{22}$$

Then, for any appropriate dimensional constant matrices $N_i (i = 1, 2, 3)$, the following term is equal to zero.

$$2\left[x^T(t)N_1 + x^T(t-\tau)N_2 + f^T(\sigma(t))N_3\right]\left[x(t) - x(t-\tau) - \int\limits_{t-\tau}^{t} \dot{x}(s)ds\right].(23)$$

On the other hand, for any appropriate dimensional constant matrices X, the following term is also equal to zero.

$$\begin{bmatrix} x(t) \\ x(t-\tau) \\ f(\sigma(t)) \end{bmatrix}^T \begin{bmatrix} \tau(X_{11}-X_{11}) & \tau(X_{12}-X_{12}) & \tau(X_{13}-X_{13}) \\ \tau(X_{12}-X_{12})^T & \tau(X_{22}-X_{22}) & \tau(X_{23}-X_{23}) \\ \tau(X_{13}-X_{13})^T & \tau(X_{23}-X_{23})^T & \tau(X_{33}-X_{33}) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ f(\sigma(t)) \end{bmatrix}.(24)$$

Calculating the derivative of $V_d(x_t)$ along the solutions of system $\mathcal{S}_0$ and adding (23) and (24) into it, one have

$$\dot{V}_d(x_t)|_{\mathcal{S}_0} = \xi^T(t)\Gamma\xi(t) - \int\limits_{t-\tau}^{t} \zeta^T(t,s)\Pi\zeta(t,s)ds, \tag{25}$$

where

$$\xi(t) = [x^T(t)\ x^T(t-\tau)\ f^T(\sigma)]^T, \zeta(t,s) = [\xi^T(t)\ \dot{x}^T(s)]^T,$$
$$\Gamma = \begin{bmatrix} \Gamma_{11} + \tau A^T ZA & \Gamma_{12} + \tau A^T ZB & \Gamma_{13} + \tau A^T ZD \\ \Gamma_{12}^T + \tau B^T ZA & \Gamma_{22} + \tau B^T ZB & \Gamma_{23} + \tau B^T ZD \\ \Gamma_{13}^T + \tau D^T ZA & \Gamma_{23}^T + \tau D^T ZB & \Gamma_{33} + \tau D^T ZD \end{bmatrix},$$

and $\Gamma_{ij}(i = 1, 2, 3; i \leq j \leq 3)$ are defined in (19) and Π is defined in (20).

In addition, the conditions (3) are equivalent to that

$$f_j(\sigma_j(t))(f_j(\sigma_j(t)) - k_j c_j^T x(t)) \leq 0, j = 1, 2, \cdots, m, \tag{26}$$

and it is easily shown that

$$\{\xi(t)|(x(t), x(t-\tau))) \neq 0 \quad \text{and } (3)\} = \{\xi(t)|\xi(t) \neq 0 \quad \text{and } (3)\}. \tag{27}$$

It now follows from (26) and (27) and applies the S-procedure, if there exist

T=diag$(t_1, t_2, \cdots, t_m) \geq 0$ such that

$$\begin{aligned} &\xi^T(t)\Gamma\xi(t) - \int_{t-\tau}^{t} \zeta^T(t,s)\Pi\zeta(t,s)ds \\ &-2\sum_{j=1}^{m} t_j f_j(\sigma_j(t))(f_j(\sigma_j(t)) - k_j c_j^T x(t)) < 0, \end{aligned} \tag{28}$$

for $\xi(t) \neq 0$, $\dot{V}_d(x_t)|_{\mathcal{S}_0} < 0$ for $(x(t), x(t-\tau)) \neq 0$ and the condition (3). (28) gives that LMIs (19) and (20) hold. So, system $\mathcal{S}_0$ is absolutely stable.

Remark 2: Delay-independent criteria may be conservative especially when the size of the delay is actually small. The comparison between these two classes of criteria will be given in the example. It will be shown that the system is not absolutely stable for any delays, but it is absolutely stable when the delay is smaller than a constant number.

In the following, Theorem 7 is easy to extended to the system with time-varying structured uncertainties using Lemma 4.

Theorem 8 *Given a scalar $\tau > 0$, system $\mathcal{S}_1$ is robustly absolutely stable if there exist $P = P^T > 0$, $Q = Q^T > 0$, $Z = Z^T \geq 0$, $X = X^T = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0$, Λ=diag $(\lambda_1, \lambda_2, \cdots, \lambda_m) \geq 0$, T=diag$(t_1, t_2, \cdots, t_m) \geq 0$, any matrices $N_i (i = 1, 2, 3)$ and a scalar $\varepsilon > 0$ such that the following LMIs (29) and (20) hold.*

$$\begin{bmatrix} \Gamma_{11} + \varepsilon E_a^T E_a & \Gamma_{12} + \varepsilon E_a^T E_b & \tilde{\Gamma}_{13} & \tau A^T Z & PH \\ \Gamma_{12}^T + \varepsilon E_b^T E_a & \Gamma_{22} + \varepsilon E_b^T E_b & \tilde{\Gamma}_{23} & \tau B^T Z & 0 \\ \tilde{\Gamma}_{13}^T & \tilde{\Gamma}_{23}^T & \tilde{\Gamma}_{33} & \tau D^T Z & \Lambda C^T H \\ \tau Z A & \tau Z B & \tau Z D & -\tau Z & \tau Z H \\ H^T P & 0 & H^T C \Lambda & \tau H^T Z & -\varepsilon I \end{bmatrix} < 0. \tag{29}$$

where

$$\begin{aligned} \tilde{\Gamma}_{13} &= \Gamma_{13} + CKT + \varepsilon E_a^T E_d, \\ \tilde{\Gamma}_{23} &= \Gamma_{23} + \varepsilon E_b^T E_d, \\ \tilde{\Gamma}_{33} &= \Gamma_{33} - 2T + \varepsilon E_d^T E_d, \end{aligned}$$

and $\Gamma_{ij} (i = 1, 2, 3; i \leq j \leq 3)$ are defined in (19).

5 Example

Consider system $\mathcal{S}_1$ with

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \ B = \begin{bmatrix} -0.5 & -0.1 \\ 0.1 & -0.5 \end{bmatrix}, \ D = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \ \ C = I,$$

and $\Delta A(t)$, $\Delta B(t)$ and $\Delta D(t)$ are uncertain matrices satisfying

$$\|\Delta A(t)\| \leq 0.2, \quad \|\Delta B(t)\| \leq 0.05, \quad \|\Delta D(t)\| \leq 0.05,$$

the above system is of the form of (4)-(5) with

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ E_a = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ E_b = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \ E_d = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix},$$

assume that $k_1 = 1, k_2 = 2.23$. Since $m = 2$, we have

$$D_1^2 = \{\mathrm{diag}(0,0)\}, \ D_2^2 = \{\mathrm{diag}(0,0), \mathrm{diag}(k_1, 0)\}.$$

By solving LMIs (19), we obtain that

$$P = \begin{bmatrix} 16.4678 & -9.3711 \\ -9.3711 & 29.0463 \end{bmatrix}, \ Q = \begin{bmatrix} 9.2568 & -6.1537 \\ -6.1537 & 27.4173 \end{bmatrix},$$
$$\lambda_1 = 0.3889, \quad \lambda_2 = 28.5416.$$

Thus, system $\mathcal{S}_1$ is robustly absolutely stable.

Moreover, LMI (13) is not true while $k_1 = 1, k_2 = 2.09$ in Theorem 5 if the S-procedure is directly adopted to examine the robust absolute stability, which indicates that it is conservative that S-procedure is directly adopted for the uncertain systems with multiple non-linearities.

In addition, we set $k_1 = 1, k_2 = 3$, then LMIs (16) are not true such that the Lyapunov functional in the extended Lur'e form to guarantee robust absolute stability of system $\mathcal{S}_1$ can't be found, but it follows from Theorem 8 that system $\mathcal{S}_1$ is robustly absolutely stable while $\tau \leq 1.5789$.

6 Conclusion

This paper presents the necessary and sufficient conditions for the existence of a Lyapunov functional in the extended Lur'e form with negative definite derivative to guarantee the robust absolute stability for delay Lur'e control systems with multiple non-linearities and converts the existence problem to a simple of solving a set of LMIs. Moreover, some delay-dependent criteria are derived for the absolute stability or robust absolute stability.

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