

Improved Bounded-Real-Lemma Representation and H_∞ Control of Systems with Polytopic Uncertainties

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Abstract—This paper concerns the problem of the bounded-real-lemma representation and H_∞ control of linear systems with real convex polytopic uncertainties. In order to use a parameter-dependent Lyapunov function for a system with polytopic uncertainties, the derivative term for the state, which is in the derivative of the Lyapunov function, is reserved; and free weighting matrices are used to express the relationship between the terms of the system equation. This yields a new LMI approach to bounded-real-lemma representation. In addition, this method is extended to the design of a state-feedback controller that solves the H_∞ control problem. Numerical examples demonstrate that the proposed method is effective and is an improvement over previous ones.

Index Terms—bounded-real-lemma, H_∞ control, parameter-dependent Lyapunov function, polytopic uncertainty, linear matrix inequality.

I. INTRODUCTION

The construction of a Lyapunov function is a basic problem in system analysis and synthesis. Over the past few years, a considerable number of studies have been devoted to systems with polytopic uncertainties. Many of them are based on the concept of quadratic stability, which attempts to find a single quadratic Lyapunov function (e.g., [1], [2]). Recent investigations have shown that a parameter-dependent Lyapunov function can overcome the conservativeness arising from the use of a single quadratic Lyapunov function for continuous linear uncertain systems [3]–[8], time-delay uncertain systems [9]–[12], and discrete uncertain systems [13]–[15]. P.J. de Oliveira *et al.* (2002) made a numerical comparison and assessed the conservativeness of quadratic stability conditions [7]. Of note is the fact that Shaked (2001) derived stability criteria and a bounded-real-lemma (BRL) representation for linear systems with real convex polytopic uncertainties [6]; and Jia (2003) [8] found alternative proofs to the theorems in [6]. In addition, their method was extended to the problem of H_∞ control. However, their BRL representation was not expressed solely in terms of a linear matrix inequality (LMI) [16].

This paper presents a simple technique for the BRL representation of a linear system. In the derivative of the Lyapunov function for a system with constant coefficient matrices, the term $\dot{x}(t)$ is reserved, and the relationship between the terms

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of the system equation is expressed by free weighting matrices. So, the Lyapunov matrices (the matrices in the Lyapunov function) do not contain any product terms involving the dynamic matrices (the matrices in the dynamic equation of the system). These features enable a parameter-dependent Lyapunov function to be used for a system with polytopic uncertainties. Moreover, this idea is extended to solve the problem of H_∞ state-feedback control. Numerical examples demonstrate that the method presented in this paper is a significant improvement over previous ones.

II. MAIN RESULTS

Consider the linear system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + B_\omega \omega(t) + Bu(t), \\ z(t) = Cx(t) + D_\omega \omega(t) + Du(t), \\ x(0) = 0, \end{cases} \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the control input, $\omega(t) \in \mathcal{L}_2^q[0, \infty)$ is an exogenous disturbance, and $z(t) \in \mathcal{R}^p$ is the controlled error. The matrices A , B_ω , B , C , D_ω , and D are constant matrices of appropriate dimensions. For a given scalar $\gamma > 0$, the performance of the system is defined to be

$$J(\omega) = \int_0^\infty (z^T z - \gamma^2 \omega^T \omega) ds. \quad (2)$$

The problem is to find a state-feedback gain, $K \in \mathcal{R}^{m \times n}$, in the control law

$$u(t) = Kx(t) \quad (3)$$

such that $J(\omega) < 0$ for all non-zero $\omega(t) \in \mathcal{L}_2^q[0, \infty)$.

First, we give a new form of BRL representation for $u(t) = 0$. Next, we extend it to a system with polytopic uncertainties. Then, we use it to solve the problem of designing an H_∞ state-feedback controller.

Theorem 1: Consider the system Σ with $u(t) = 0$. For a given scalar $\gamma > 0$, $J(\omega) < 0$ holds for all nonzero $\omega(t) \in \mathcal{L}_2^q[0, \infty)$ if there exist a symmetric positive definite matrix $P = P^T > 0$ and any appropriately dimensioned matrices T_j ($j = 1, 2$) such that the following LMI holds:

$$\Xi = \begin{bmatrix} -A^T T_1^T - T_1 A & -T_1 B_\omega & \Xi_{13} & C^T \\ -B_\omega^T T_1^T & -\gamma^2 I & -B_\omega^T T_2^T & D_\omega^T \\ \Xi_{13}^T & -T_2 B_\omega & T_2 + T_2^T & 0 \\ C & D_\omega & 0 & -I \end{bmatrix} < 0, \quad (4)$$

where $\Xi_{13} = P + T_1 - A^T T_2^T$.

Proof: Choose a Lyapunov function candidate to be

$$V(x) := x^T(t)Px(t), \quad (5)$$

where $P = P^T > 0$ needs to be determined. According to (1), for any appropriately dimensioned matrices T_j ($j = 1, 2$), we have

$$2[x^T(t)T_1 + \dot{x}^T(t)T_2][\dot{x}(t) - Ax(t) - B_\omega\omega(t)] = 0. \quad (6)$$

Calculating the derivative of $V(x)$ along the solutions of Σ and adding Equation (6) to it yields

$$\begin{aligned} & \dot{V}(x) + z^T z - \gamma^2 \omega^T \omega \\ &= 2x^T(t)P\dot{x}(t) \\ & \quad + [Cx(t) + D_\omega\omega(t)]^T [Cx(t) + D_\omega\omega(t)] \\ & \quad - \gamma^2 \omega^T(t)\omega(t) \\ & \quad + 2[x^T(t)T_1 + \dot{x}^T(t)T_2][\dot{x}(t) - Ax(t) - B_\omega\omega(t)] \\ &:= \zeta^T(t)\Phi\zeta(t), \end{aligned} \quad (7)$$

where

$$\zeta(t) := [x^T(t) \ \omega^T(t) \ \dot{x}^T(t)]^T, \\ \Phi = \begin{bmatrix} -A^T T_1^T - T_1 A + C^T C & -T_1 B_\omega + C^T D_\omega \\ -B_\omega^T T_1^T + D_\omega^T C & -\gamma^2 I + D_\omega^T D_\omega \\ P + T_1^T - T_2 A & -T_2 B_\omega \\ P + T_1 - A^T T_2^T & \\ -B_\omega^T T_2^T & \\ T_2 + T_2^T & \end{bmatrix}.$$

Clearly, if $\Phi < 0$, then $\dot{V}(x) + z^T z - \gamma^2 \omega^T \omega < 0$ for any $\zeta(t) \neq 0$. The initial condition $x(0) = 0$ implies that $J(\omega) < 0$ [16]. Applying the Schur complement [16] shows that $\Phi < 0$ is equivalent to $\Xi < 0$. So, $J(\omega) < 0$ for all nonzero $\omega(t) \in \mathcal{L}_2^q[0, \infty)$ if LMI (4) holds. ■

Remark 1: A well-known condition for $J(\omega) < 0$ for all nonzero $\omega(t) \in \mathcal{L}_2^q[0, \infty)$ [16] is that there exists a symmetric positive definite matrix $P = P^T > 0$ such that

$$\begin{bmatrix} PA + A^T P & PB_\omega & C^T \\ B_\omega^T P & -\gamma^2 I & D_\omega^T \\ C & D_\omega & -I \end{bmatrix} < 0. \quad (8)$$

This condition is equivalent to the one in Theorem 1. In fact, left-multiplying the third row of Ξ in (4) by A^T or B_ω^T and adding it to the first or second row, and right-multiplying the third column of Ξ by A or B_ω and adding it to the first or second column yields the following LMI:

$$\begin{bmatrix} PA + A^T P & PB_\omega & P + T_1 - A^T T_2^T & C^T \\ B_\omega^T P & -\gamma^2 I & -B_\omega^T T_2^T & D_\omega^T \\ P + T_1^T - T_2 A & -T_2 B_\omega & T_2 + T_2^T & 0 \\ C & D_\omega & 0 & -I \end{bmatrix} < 0. \quad (9)$$

It is clear that LMI (8) is feasible if LMI (9) is feasible. On the other hand, a feasible solution, P , of LMI (8) is also a feasible solution of LMI (9). For example, by setting $T_1 = -P$, LMI (9) is transformed into

$$\begin{bmatrix} PA + A^T P & PB_\omega & -A^T T_2^T & C^T \\ B_\omega^T P & -\gamma^2 I & -B_\omega^T T_2^T & D_\omega^T \\ -T_2 A & -T_2 B_\omega & T_2 + T_2^T & 0 \\ C & D_\omega & 0 & -I \end{bmatrix} < 0. \quad (10)$$

If T_2 is chosen to be $-\delta I$, where δ is a sufficiently small positive scalar, then P is a solution of LMI (10) if it is a feasible solution of LMI (8).

The importance of Theorem 1 is that it separates P from A , B_ω , C , and D_ω ; that is, there are no terms containing the product of P and any of them. This enables a new robust BRL to be derived for a system with polytopic uncertainties by using a parameter-dependent Lyapunov function.

Assuming that the matrices A , B_ω , C , and D_ω of Σ are known to lie within the polytopic uncertainties, Ω , we have

$$\begin{aligned} [A \ B_\omega \ C \ D_\omega] \in \Omega &:= \{[A(\xi) \ B_\omega(\xi) \ C(\xi) \ D_\omega(\xi)] \\ &= \sum_{i=1}^p \xi_i [A_i \ B_{\omega,i} \ C_i \ D_{\omega,i}], \sum_{i=1}^p \xi_i = 1, \xi_i \geq 0\}, \end{aligned} \quad (11)$$

where A_i , $B_{\omega,i}$, C_i , and $D_{\omega,i}$ ($i = 1, \dots, p$) are constant matrices with appropriate dimensions; and ξ_i ($i = 1, \dots, p$) are time-invariant uncertainties. Theorem 1 is extended to a system with polytopic uncertainties by employing a parameter-dependent Lyapunov function as follows:

Theorem 2: Consider the system Σ with $u(t) = 0$ and polytopic uncertainties (11). For a given scalar $\gamma > 0$, $J(\omega) < 0$ holds for all nonzero $\omega(t) \in \mathcal{L}_2^q[0, \infty)$ if there exist symmetric positive definite matrices $P_i = P_i^T > 0$ ($i = 1, \dots, p$) and any appropriately dimensioned matrices T_j ($j = 1, 2$) such that the following LMIs hold for $i = 1, \dots, p$:

$$\Psi_i = \begin{bmatrix} \Psi_{11}^{(i)} & -T_1 B_{\omega,i} & \Psi_{13}^{(i)} & C_i^T \\ -B_{\omega,i}^T T_1^T & -\gamma^2 I & -B_{\omega,i}^T T_2^T & D_{\omega,i}^T \\ (\Psi_{13}^{(i)})^T & -T_2 B_{\omega,i} & T_2 + T_2^T & 0 \\ C_i & D_{\omega,i} & 0 & -I \end{bmatrix} < 0, \quad (12)$$

where

$$\begin{aligned} \Psi_{11}^{(i)} &= -A_i^T T_1^T - T_1 A_i, \\ \Psi_{13}^{(i)} &= P_i + T_1 - A_i^T T_2^T. \end{aligned}$$

Proof: Choose a parameter-dependent Lyapunov function candidate to be

$$V_p(x) := \sum_{i=1}^p x^T(t)\xi_i P_i x(t) := x^T(t)P(\xi)x(t), \quad (13)$$

where $P_i = P_i^T > 0$ ($i = 1, \dots, p$) need to be determined.

Equation (6), in which T_j ($j = 1, 2$) are any appropriately dimensioned matrices, is employed in the following calculation:

$$\begin{aligned} & \dot{V}_p(x) + z^T z - \gamma^2 \omega^T \omega \\ &= 2x^T(t)P(\xi)\dot{x}(t) \\ & \quad + [C(\xi)x(t) + D_\omega(\xi)\omega(t)]^T [C(\xi)x(t) + D_\omega(\xi)\omega(t)] \\ & \quad - \gamma^2 \omega^T(t)\omega(t) \\ & \quad + 2[x^T(t)T_1 + \dot{x}^T(t)T_2][\dot{x}(t) - A(\xi)x(t) - B_\omega(\xi)\omega(t)] \\ &:= \zeta^T(t)\Psi(\xi)\zeta(t), \end{aligned} \quad (14)$$

where $\zeta(t)$ is defined in (7), and $\Psi(\xi)$ has the same structure as Φ in (7) but with A , B_ω , C , and D_ω replaced with $A(\xi)$, $B_\omega(\xi)$, $C(\xi)$, and $D_\omega(\xi)$, respectively. If $\Psi(\xi) < 0$, then $\dot{V}_p(x) + z^T z - \gamma^2 \omega^T \omega < 0$ for any $\zeta(t) \neq 0$. And the initial condition $x(0) = 0$ implies that $J(\omega) < 0$ [16]. On the other hand, applying the Schur complement shows that

$\Psi_i < 0$ ($i = 1, \dots, p$) ensures that $\Psi(\xi) < 0$. So, $J(\omega) < 0$ for all nonzero $\omega \in \mathcal{L}_2^q[0, \infty)$ holds if LMIs (12) hold. ■

Remark 2: A parameter-dependent Lyapunov function is employed in Theorem 2. This theorem is an extension of Theorem 1, which is equivalent to the well-known condition (8). Lemmas 2.1 and 2.2 in [6] extend the condition (8) to yield a BRL representation of a system with polytopic uncertainties. This overcomes the conservativeness of quadratic stability conditions, but these Lemmas are only sufficient conditions for (8). In particular, the LMI conditions in [6] contain a scalar tuning parameter, ϵ , the choice of which may lead to conservativeness. In contrast, Theorem 2 is based on LMIs (12), which do not contain any tuning parameters.

Theorem 2 can be used to design an H_∞ state-feedback controller for a system with polytopic uncertainties as follows:

Theorem 3: Consider the system Σ with polytopic uncertainties (11). For given scalars λ and $\gamma > 0$, the state-feedback control law (3) guarantees that $J(\omega) < 0$ for all nonzero $\omega(t) \in \mathcal{L}_2^q[0, \infty)$ if there exist symmetric positive definite matrices $X_i = X_i^T > 0$ ($i = 1, \dots, p$) and any appropriately dimensioned matrices S and V such that the following LMIs hold for $i = 1, \dots, p$:

$$\Gamma_i = \begin{bmatrix} \Gamma_{11}^{(i)} & -B_{\omega,i} & \Gamma_{13}^{(i)} & \Gamma_{14}^{(i)} \\ -B_{\omega,i}^T & -\gamma^2 I & -\lambda B_{\omega,i}^T & D_{\omega,i}^T \\ (\Gamma_{13}^{(i)})^T & -\lambda B_{\omega,i} & \lambda(S + S^T) & 0 \\ (\Gamma_{14}^{(i)})^T & D_{\omega,i} & 0 & -I \end{bmatrix} < 0, \quad (15)$$

where

$$\begin{aligned} \Gamma_{11}^{(i)} &= -SA_i^T - A_i S^T - BV - V^T B^T, \\ \Gamma_{13}^{(i)} &= X_i + S^T - \lambda SA_i^T - \lambda V^T B^T, \\ \Gamma_{14}^{(i)} &= SC_i^T + V^T D^T. \end{aligned}$$

Moreover, the control law is $u(t) = VS^{-T}x(t)$.

Proof: In (3), we replace $A(\xi)$ and $C(\xi)$ in $\Psi(\xi)$ in (14) with $A(\xi) + BK$ and $C(\xi) + DK$, respectively, and set $T_1 = T$ and $T_2 = \lambda T$. Based on the fact that $T_2 + T_2^T$ in $\Psi(\xi)$ is negative definite, clearly T is nonsingular. Then, if we premultiply $\text{diag}(T^{-1}, I, T^{-1})$ by $\Psi(\xi)$ in (14), postmultiply $\text{diag}(T^{-T}, I, T^{-T})$ by $\Psi(\xi)$, let $S = T^{-1}$, $X_i = SP_i S^T$ ($i = 1, \dots, p$) and $K = VS^{-T}$, and apply the Schur complement, then we obtain (15). ■

Remark 3: Theorem 3 contains a tuning parameter, λ ; and the optimum value can be ascertained by the approach in Remark 5 of [10]. Furthermore, a numerical optimization algorithm, such as `fminsearch` in the Optimization Toolbox Ver. 2.2 of Matlab 6.5, can provide a numerical solution to this problem.

III. EXAMPLES

In this section, the two examples discussed in [6] are employed to demonstrate the effectiveness of the method presented in this paper and the improvement over previous ones.

Example 1: [6] Consider the uncertain system Σ with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1+g & -1-g \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C &= [1 \quad -2], \quad D_\omega = [0], \end{aligned}$$

where g is an uncertain parameter that is known to lie in the interval $[-g_1, g_1]$. A can be rewritten as

$$\begin{aligned} A &= \xi \begin{bmatrix} 0 & 1 \\ -1+g_1 & -1-g_1 \end{bmatrix} \\ &+ (1-\xi) \begin{bmatrix} 0 & 1 \\ -1-g_1 & -1+g_1 \end{bmatrix}. \end{aligned}$$

For $g_1 = 0.3777$, the methods in [6], [16] yield values of 5 and 4.488, respectively, for the performance, γ . In contrast, Theorem 2 yields a minimum γ of 3.4963, which is clearly much better.

Example 2: [2], [6] Consider the satellite system in [2], which has the following state-space representation [6]:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} &= \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mathcal{T}. \end{aligned} \quad (16)$$

The controlled error, z , is given by

$$z = \begin{bmatrix} \theta_2 \\ 0.01\mathcal{T} \end{bmatrix}; \quad (17)$$

and \mathcal{T} is the control input. The parameters of the system are assumed to be $J_1 = J_2 = 1$, $k \in [0.09 \ 0.4]$, and $f \in [0.0038 \ 0.04]$. Theorem 3 with $\lambda = 0.12$ yields a value of 1.117 for the level of attenuation, γ , of the following controller:

$$K = -[219.8 \ 2356.5 \ 61.3 \ 3058.0].$$

However, the minimum γ is as large as 1.557 in [16], which uses the quadratic stabilization method, and 1.478 in [6].

IV. CONCLUSION

This paper describes a new technique for BRL representation and its application to the H_∞ control of linear systems with polytopic uncertainties. First, a simple criterion is presented for a system with constant coefficient matrices. It employs free weighting matrices to take the relationship between the terms of the system equation into account. With this treatment, no terms appear that involve the product of the Lyapunov matrices and the dynamic matrices, which makes it easy to extend the treatment to a system with polytopic uncertainties. This method is further extended to the design of an H_∞ state-feedback controller. Numerical examples demonstrate that the methods described in this paper are very effective and are a significant improvement over previous ones.

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