Delay-dependent exponential stability of delayed neural networks with time-varying delay

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***Abstract*— In this paper, free-weighting matrices are employed to express the relationship between the terms in the Leibniz-Newton formula; and based on that relationship, a new delay-dependent exponential-stability criterion is derived for delayed neural networks with a time-varying delay. Two numerical examples demonstrate the improvement this method provides over existing ones.**

***Index Terms*— neural networks, delay-dependent criterion, exponential stability, linear matrix inequality (LMI), free-weighting matrix approach.**

I. Introduction

Neural networks have been extensively studied over the past few decades and have found application in a variety of areas, such as pattern recognition, associative memory, and combinatorial optimization. These applications strongly depend on the dynamic behavior of the network. In recent years, considerable effort has been devoted to analyzing the stability of neural networks without a time delay.

In reality, however, the dynamics of a neural network often involves time delays due, for example, to the finite switching speed of amplifiers in electronic neural networks, or to the finite signal propagation time in biological networks. Recently, the stability of delayed neural networks has received considerable attention (see e.g. [1]–[26]). The criteria derived in these papers are based on various types of stability, such as asymptotic stability, complete stability, absolute stability, and exponential stability. As pointed out in [13], the property of exponential stability is particularly important when the exponential convergence rate is used to determine the speed of neural computations. Thus, in general, it is not only theoretically interesting but also of practical importance to determine the exponential stability of, and to estimate the exponential convergence rate of, dynamic neural networks. Accordingly, a great number of sufficient conditions guaranteeing the global exponential stability of delayed neural networks with constant and time-varying delays have been derived [1], [4], [13], [19], [20], [24], [27]–[32]. Among them, delay-dependent exponential-stability criteria have attracted much attention recently [13], [24] because delay-dependent criteria make use of information on the length of delays, and are less conservative than delay-independent ones. However, some negative terms in the derivative of the Lyapunov functional tend to be ignored when delay-dependent stability criteria are derived [13], [24]. This may lead to considerable conservativeness. It is also worth mentioning that, in those papers, the restriction that the derivative of a time-varying delay be less than 1 is imposed on stability criteria for neural networks with a time-varying delay. Recently, a free-weighting-matrix approach was proposed [33]–[36] in which free weighting matrices are employed to express the relationship between the terms of the Leibniz-Newton formula; and all the negative terms in the derivative of the Lyapunov functional are retained. This approach avoids the restriction on the derivative of a time-varying delay.

In this paper, the free-weighting-matrix approach is employed to derive an LMI-based delay-dependent exponential-stability criterion for neural networks with a time-varying delay. Unlike existing ones, this criterion allows the derivative of a time-varying delay to take any value. And since the negative terms in the derivative of the Lyapunov functional are retained, this criterion is less conservative than existing ones. Numerical examples demonstrate the effectiveness of this method.

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II. System Description

Consider the following delayed neural network:

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) + u, \quad (1)$$

where $x(\cdot) = [x_1(\cdot),\ x_2(\cdot),\ \cdots, x_n(\cdot)]^T \in \mathcal{R}^n$ is the neuron state vector; $g(x(\cdot)) = [g_1(x_1(\cdot)),\ g_2(x_2(\cdot)),\ \cdots,\ g_n(x_n(\cdot))]^T \in \mathcal{R}^n$ denotes the neuron activation function; $u = [u_1,\ u_2,\ \cdots,\ u_n]^T \in \mathcal{R}^n$ is a constant input vector; $C = \mathrm{diag}\{c_1,\ c_2,\ \cdots,\ c_n\}$ is a diagonal matrix with $c_i > 0$; and A and B are the connection weight matrix and the delayed connection weight matrix, respectively, The time delay, $\tau(t)$, is a time-varying differentiable function that satisfies

$$0 \le \tau(t) \le \bar{\tau}, \quad (2)$$

$$\dot{\tau}(t) \le \mu, \quad (3)$$

where μ is a constant. In addition, it is assumed that each neuron activation function in (1), $g_j(\cdot),\ j = 1, 2, \cdots, n$, satisfies the following condition:

$$0 \le \frac{g_j(x) - g_j(y)}{x - y} \le L_j,\ \ \forall\, x, y \in \mathcal{R},\ x \ne y,\ j = 1, 2, \cdots, n, \quad (4)$$

where $L_j,\ j=1,2,\cdots,n$ are positive constants.

In the following, the equilibrium point $x^*=[x_1^*,\ x_2^*,\ \cdots,\ x_n^*]^T$ of (1) is first shifted to the origin by the transformation $z(\cdot)=x(\cdot)-x^*$, which converts the system to the following form:

$$\dot{z}(t)=-Cz(t)+Af(z(t))+Bf(z(t-\tau(t))), \quad (5)$$

where $z(\cdot)=[z_1(\cdot),\ z_2(\cdot),\ \cdots,\ z_n(\cdot)]^T$ is the state vector of the transformed system; $f(z(\cdot))=[f_1(z_1(\cdot)),\ f_2(z_2(\cdot)),\ \cdots,\ f_n(z_n(\cdot))]^T$; and $f_j(z_j(\cdot))=g_j(z_j(\cdot)+z_j^*)-g_j(z_j^*),\ j=1,2,\cdots,n$. Note that the functions $f_j(\cdot)\ (j=1,2,\cdots,n)$ satisfy the following condition:

$$0\le\frac{f_j(z_j)}{z_j}\le L_j,\quad f_j(0)=0,\quad \forall z_j\neq 0,\ j=1,2,\cdots,n, \quad (6)$$

which is equivalent to

$$f_j(z_j)\left[f_j(z_j)-L_jz_j\right]\le 0,\quad f_j(0)=0,\quad j=1,2,\cdots,n. \quad (7)$$

The definition of exponential stability is now given.

Definition 1: The system (5) is said to be exponentially stable if there exist constants $k>0$ and $M\ge 1$ such that

$$\|z(t)\|\le M\phi e^{-kt}, \quad (8)$$

where

$$\phi=\sup_{-\bar{\tau}\le\theta\le 0}\|z(\theta)\|$$

and $\|z\|$ is the Euclidean norm of z. Furthermore, k is called the exponential convergence rate.

The following lemmas are employed to derive the new criterion.

Lemma 1: For any vectors $a,b\in\mathcal{R}^n$, the inequality

$$2a^Tb\le a^TXa+b^TX^{-1}b \quad (9)$$

holds, where X is any positive matrix (i.e. $X>0$).

Lemma 2: [24] Assuming that (6) holds, then

$$\int_v^u\left[f_i(s)-f_j(s)\right]ds\le[u-v]\left[f_i(u)-f_i(v)\right], \quad j=1,2,\cdots,n. \quad (10)$$

III. Delay-Dependent Exponential-Stability Criterion

The following delay-dependent exponential-stability criterion is obtained by employing the free-weighting-matrix approach described in [33]–[36].

Theorem 1: For given scalars $k:\ 0<k<\min c_i\ (i=1,\cdots,n)$, $\bar{\tau}\ge 0$, and d, the origin of system (5) with (6) and a time delay satisfying conditions (2) and (3) is globally exponential stable and has the exponential convergence rate k if there exist $P=P^T>0$, $Q=Q^T>0$, $W=W^T>0$, $Z=Z^T>0$, $D=\mathrm{diag}(d_1,d_2,\cdots,d_n)\ge 0$, $R=\mathrm{diag}(r_1,r_2,\cdots,r_n)\ge 0$, $S=\mathrm{diag}(s_1,s_2,\cdots,s_n)\ge 0$, and any appropriately dimensioned matrices $T_i\ (i=1,2)$ and $N=\left[N_1^T\ N_2^T\ N_3^T\ N_4^T\ N_5^T\ \right]^T$ such that the following LMI (11) is feasible:

$$\Xi=\begin{bmatrix}\Phi & \bar{\tau}N\\ \bar{\tau}N^T & -\bar{\tau}e^{-2k\bar{\tau}}Z\end{bmatrix}<0, \quad (11)$$

where

$$\Phi=\begin{bmatrix}\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15}\\ \Phi_{12}^T & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25}\\ \Phi_{13}^T & \Phi_{23}^T & \Phi_{33} & \Phi_{34} & \Phi_{35}\\ \Phi_{14}^T & \Phi_{24}^T & \Phi_{34}^T & \Phi_{44} & 0\\ \Phi_{15}^T & \Phi_{25}^T & \Phi_{35}^T & 0 & \Phi_{55}\end{bmatrix},$$

$$\begin{aligned}
\Phi_{11}&=2kP+T_1C+C^TT_1^T+N_1+N_1^T+e^{2k\bar{\tau}}Q,\\
\Phi_{12}&=P+T_1+C^TT_2^T+N_2^T,\\
\Phi_{13}&=N_3^T-N_1,\\
\Phi_{14}&=2kD-T_1A+LR+N_4^T,\\
\Phi_{15}&=-T_1B+N_5^T,\\
\Phi_{22}&=T_2+T_2^T+\bar{\tau}Z,\\
\Phi_{23}&=-N_2,\\
\Phi_{24}&=D-T_2A,\\
\Phi_{25}&=-T_2B,\\
\Phi_{33}&=-(1-\mu)Q-N_3-N_3^T,\\
\Phi_{34}&=-N_4^T,\\
\Phi_{35}&=LS-N_5^T,\\
\Phi_{44}&=e^{2k\bar{\tau}}W-2R,\\
\Phi_{55}&=-(1-\mu)W-2S,\\
L&=\mathrm{diag}(L_1,L_2,\cdots,L_n).
\end{aligned}$$

Proof: Construct the following Lyapunov-Krasovskii functional:

$$\begin{aligned}
V(z(t)) &= V_1(z(t))+V_2(z(t))+V_3(z(t)),\\
V_1(z(t)) &= e^{2kt}z^T(t)Pz(t)+2\sum_{j=1}^n d_je^{2kt}\int_0^{z_j}f_j(s)ds,\\
V_2(z(t)) &= e^{2k\bar{\tau}}\int_{t-\tau(t)}^t e^{2ks}\\
&\quad\times\left[z^T(s)Qz(s)+f^T(z(s))Wf(z(s))\right]ds,\\
V_3(z(t)) &= \int_{-\bar{\tau}}^0\int_{t+\theta}^t e^{2ks}\dot{z}^T(s)Z\dot{z}(s)dsd\theta,
\end{aligned} \quad (12)$$

where $P=P^T>0$, $Q=Q^T>0$, $W=W^T>0$, $Z=Z^T>0$, and $D=\mathrm{diag}(d_1,d_2,\cdots,d_n)\ge 0$ are to be determined.

For any appropriately dimensioned matrices $T_i\ (i=1,2)$, the following holds:

$$\begin{aligned}
0=2e^{2kt}&\left[z^T(t)T_1+\dot{z}^T(t)T_2\right]\\
&\times\left[\dot{z}(t)+Cz(t)-Af(z(t))-Bf(z(t-\tau(t)))\right].
\end{aligned} \quad (13)$$

Using the Leibniz-Newton formula, for any appropriately dimensioned matrix N, the following is also true:

$$0=2e^{2kt}\zeta^T(t)N\cdot\left[z(t)-z(t-\tau(t))-\int_{t-\tau(t)}^t\dot{z}(s)ds\right], \quad (14)$$

where

$$\zeta(t)=[z^T(t),\dot{z}^T(t),z^T(t-\tau(t)),f^T(z(t)),\ f^T(z(t-\tau(t)))]^T.$$

In addition, for any semi-positive definite matrix

$$X=X^T=\begin{bmatrix}X_{11} & X_{12} & X_{13} & X_{14} & X_{15}\\ X_{12}^T & X_{22} & X_{23} & X_{24} & X_{25}\\ X_{13}^T & X_{23}^T & X_{33} & X_{34} & X_{35}\\ X_{14}^T & X_{24}^T & X_{34}^T & X_{44} & X_{45}\\ X_{15}^T & X_{25}^T & X_{35}^T & X_{45}^T & X_{55}\end{bmatrix}\ge 0,$$

the following holds:

$$0 \leq e^{2kt}\left[\bar{\tau}\zeta^T(t)X\zeta(t) - \int_{t-\tau(t)}^{t} \zeta^T(t)X\zeta(t)ds\right]. \quad (15)$$

It is clear from (7) that

$$f_j(z_j(t))\left[f_j(z_j(t)) - L_j z_j(t)\right] \leq 0, \; j = 1, 2, \cdots, n, \quad (16)$$

and

$$f_j(z_j(t-\tau(t))) \cdot [f_j(z_j(t-\tau(t))) - L_j z_j(t-\tau(t))] \leq 0, \quad j = 1, 2, \cdots, n. \quad (17)$$

So, for any $R = \text{diag}(r_1, r_2, \cdots, r_n) \geq 0$, and $S = \text{diag}(s_1, s_2, \cdots, s_n) \geq 0$, it follows from (16) and (17) that

$$\begin{aligned} 0 &\leq -2e^{2kt}\sum_{j=1}^{n} r_j f_j(z_j(t))\left[f_j(z_j(t)) - L_j z_j(t)\right] \\ &\quad -2e^{2kt}\sum_{j=1}^{n} s_j f_j(z_j(t-\tau(t))) \\ &\quad \times [f_j(z_j(t-\tau(t))) - L_j z_j(t-\tau(t))] \\ &= 2e^{2kt}\left[z^T(t)LRf(z(t)) - f^T(z(t))Rf(z(t)) \right. \\ &\quad +z^T(t-\tau(t))LSf(z(t-\tau(t))) \\ &\quad \left. -f^T(z(t-\tau(t)))Sf(z(t-\tau(t)))\right]. \end{aligned} \quad (18)$$

Calculating the derivatives of $V_i(z(t))$ $(i = 1, 2, 3)$ along the trajectories of the system (5) yields

$$\begin{aligned} \dot{V}_1(z(t)) &= 2ke^{2kt}z^T(t)Pz(t) + 2e^{2kt}z^T(t)P\dot{z}(t) \\ &\quad +4\sum_{j=1}^{n} kd_j e^{2kt}\int_0^{z_j} f_j(s)ds \\ &\quad +2\sum_{j=1}^{n} d_j e^{2kt} f_j(z_j(t))\dot{z}_j(t) \\ &\leq 2ke^{2kt}z^T(t)Pz(t) + 2e^{2kt}z^T(t)P\dot{z}(t) \\ &\quad +4ke^{2kt}f^T(z(t))Dz(t) \\ &\quad +2e^{2kt}f^T(z(t))D\dot{z}(t), \end{aligned} \quad (19)$$

$$\begin{aligned} \dot{V}_2(z(t)) &= e^{2k\bar{\tau}}e^{2kt}\left[z^T(t)Qz(t) + f^T(z(t))Wf(z(t))\right] \\ &\quad -e^{2k\bar{\tau}}e^{2k(t-\tau(t))}(1-\dot{\tau}(t)) \\ &\quad \times \left[z^T(t-\tau(t))Qz(t-\tau(t)) \right. \\ &\quad \left. +f^T(z(t-\tau(t)))Wf(z(t-\tau(t)))\right] \\ &\leq e^{2k\bar{\tau}}e^{2kt}\left[z^T(t)Qz(t) + f^T(z(t))Wf(z(t))\right] \\ &\quad -e^{2kt}(1-\mu)\left[z^T(t-\tau(t))Qz(t-\tau(t)) \right. \\ &\quad \left. +f^T(z(t-\tau(t)))Wf(z(t-\tau(t)))\right], \end{aligned} \quad (20)$$

$$\begin{aligned} \dot{V}_3(z(t)) &= \bar{\tau}e^{2kt}\dot{z}^T(t)Z\dot{z}(t) - \int_{t-\bar{\tau}}^{t} e^{2ks}\dot{z}^T(s)Z\dot{z}(s)ds \\ &\leq \bar{\tau}e^{2kt}\dot{z}^T(t)Z\dot{z}(t) \\ &\quad -e^{2k(t-\bar{\tau})}\int_{t-\bar{\tau}}^{t} \dot{z}^T(s)Z\dot{z}(s)ds \\ &\leq \bar{\tau}e^{2kt}\dot{z}^T(t)Z\dot{z}(t) \\ &\quad -e^{2k(t-\bar{\tau})}\int_{t-\tau(t)}^{t} \dot{z}^T(s)Z\dot{z}(s)ds. \end{aligned} \quad (21)$$

Adding the terms on the right of equations (13), (14), (15), and (18) to $\dot{V}(z(t))$ yields:

$$\begin{aligned} &\dot{V}(z(t)) \\ &\leq e^{2kt}\left\{\zeta^T(t)\left[\Phi + \bar{\tau}X\right]\zeta(t) - \int_{t-\tau(t)}^{t} \eta^T(t,s)\Psi\eta(t,s)ds\right\}, \end{aligned} \quad (22)$$

where

$$\eta(t,s) = [\zeta^T(t) \;\; \dot{z}^T(s)]^T,$$
$$\Psi = \begin{bmatrix} X & N \\ N^T & e^{-2k\bar{\tau}}Z \end{bmatrix},$$

and Φ is defined in (11). If $\Phi + \bar{\tau}X < 0$ and $\Psi \geq 0$, then $\dot{V}(z(t)) < 0$ for any $\zeta(t) \neq 0$. Let $X = e^{2k\bar{\tau}}NZ^{-1}N^T$, which ensures that $X \geq 0$ and $\Psi \geq 0$. In this case, $\Phi + \bar{\tau}X < 0$ is equivalent to $\Xi < 0$, according to the Schur complement.

It follows from $\dot{V}(z(t)) < 0$ that

$$V(z(t)) \leq V(z(0)). \quad (23)$$

However, applying Lemma 2 yields

$$\begin{aligned} V(z(0)) &= z^T(0)Pz(0) + 2\sum_{j=1}^{n} d_j\int_0^{z_j(0)} f_j(s)ds \\ &\quad +e^{2k\bar{\tau}}\int_{-\tau(0)}^{0} e^{2ks} \\ &\quad \times \left[z^T(s)Qz(s) + f^T(z(s))Wf(z(s))\right]ds \\ &\quad +\int_{-\bar{\tau}}^{0}\int_{\theta}^{0} e^{2ks}\dot{z}^T(s)Z\dot{z}(s)dsd\theta \\ &\leq \lambda_{max}(P)\|\phi\|^2 + 2\sum_{j=1}^{n} d_j z_j(0)f_j(z_j(0))ds \\ &\quad +e^{2k\bar{\tau}}\lambda_{max}(Q)\int_{-\tau(0)}^{0} z^T(s)z(s)ds \\ &\quad +e^{2k\bar{\tau}}\lambda_{max}(W)\int_{-\tau(0)}^{0} f^T(z(s))f(z(s))ds \\ &\quad +\lambda_{max}(Z)\int_{-\bar{\tau}}^{0}\int_{\theta}^{0} \dot{z}^T(s)\dot{z}(s)dsd\theta. \end{aligned} \quad (24)$$

It follows from Lemma 1 that

$$\begin{aligned} &\dot{z}^T(s)\dot{z}(s) \\ &= \left[-Cz(s) + Af(z(s)) + Bf(z(s-\tau(s)))\right]^T \\ &\quad \times \left[-Cz(s) + Af(z(s)) + Bf(z(s-\tau(s)))\right] \\ &= z^T(s)C^TCz(s) + f^T(z(s))A^TAf(z(s)) \\ &\quad +f^T(z(s-\tau(s)))B^TBf(z(s-\tau(s))) \\ &\quad -2z^Tz(s)C^TAf(z(s)) - 2z^Tz(s)C^TBf(z(s-\tau(s))) \\ &\quad +2f^T(z(s))A^TBf(z(s-\tau(s))) \\ &\leq 3\left[z^T(s)C^TCz(s) + f^T(z(s))A^TAf(z(s)) \right. \\ &\quad \left. +f^T(z(s-\tau(s)))B^TBf(z(s-\tau(s)))\right] \\ &\leq 3\left[\lambda_{max}(C^TC) + \lambda_{max}(A^TA)\lambda_{max}(L^2) \right. \\ &\quad \left. +\lambda_{max}(B^TB)\lambda_{max}(L^2)\right]\|\phi\|^2. \end{aligned} \quad (25)$$

Thus,

$$\begin{aligned} V(z(0)) \leq\ & \lambda_{max}(P)\|\phi\|^2 + 2\lambda_{max}(DL)\|\phi\|^2 \\ & + \bar{\tau}e^{2k\bar{\tau}}\lambda_{max}(Q)\|\phi\|^2 \\ & + \bar{\tau}e^{2k\bar{\tau}}\lambda_{max}(W)\lambda_{max}(L^2)\|\phi\|^2 \\ & + 3\bar{\tau}^2\lambda_{max}(Z)\left[\lambda_{max}(C^TC)\right. \\ & + \lambda_{max}(A^TA)\lambda_{max}(L^2) \\ & \left. + \lambda_{max}(B^TB)\lambda_{max}(L^2)\right]\|\phi\|^2 \\ =\ & \Lambda\|\phi\|^2, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \Lambda =\ & \lambda_{max}(P) + 2\lambda_{max}(DL) + \bar{\tau}e^{2k\bar{\tau}}\lambda_{max}(Q) \\ & + \bar{\tau}e^{2k\bar{\tau}}\lambda_{max}(W)\lambda_{max}(L^2) \\ & + 3\bar{\tau}^2\lambda_{max}(Z)\left[\lambda_{max}(C^TC)\right. \\ & \left. + \lambda_{max}(A^TA)\lambda_{max}(L^2) + \lambda_{max}(B^TB)\lambda_{max}(L^2)\right]. \end{aligned}$$

On the other hand,

$$V(z(t)) \geq e^{2kt}z^T(t)Pz(t) \geq e^{2kt}\lambda_{min}(P)\|z(t)\|^2. \tag{27}$$

Therefore,

$$\|z(t)\| \leq \sqrt{\frac{\Lambda}{\lambda_{min}(P)}}\|\phi\|e^{-kt}. \tag{28}$$

From Definition 1, (5) is exponentially stable and has the exponential convergence rate k. This completes the proof. ■

Remark 1: In the derivation of the delay-dependent exponential-stability criteria in [24], the negative terms in the derivative of $V_3(t)$ are ignored, which may lead to conservatism. In contrast, the proof of Theorem 1 shows that the negative term $-e^{2k(t-\bar{\tau})}\int_{t-\tau(t)}^{t}\dot{z}^T(s)Z\dot{z}(s)ds$ in $\dot{Z}_3(z(t))$ is retained. The free-weighting-matrix approach is employed to handle it and to derive the delay-dependent exponential-stability criterion.

IV. Numerical Examples

This section provides two numerical examples that demonstrate the effectiveness of the criterion presented in this paper.

Example 1: Consider the delayed neural network (1) with

$$\begin{aligned} & C = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \\ & B = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},\ L_1 = L_2 = 1,\ \tau(t) = 1. \end{aligned} \tag{29}$$

When $k = 0.25$ (note that the exponential convergence rate is $k/2$ in [24]), Theorem 1 in [24] shows the system to be globally exponentially stable; but Theorem 3 in [18] fails to verify that. However, Theorem 1 in this paper in combination with the bisection search method shows the system to be globally exponentially stable, even for $k = 1.15$.

Example 2: Consider the delayed neural network (1) with

$$\begin{aligned} & C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \\ & B = \begin{bmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix},\ L_1 = L_2 = 1. \end{aligned} \tag{30}$$

It is assumed that $\tau(t) = \frac{1}{2}\sin^2 t$. So, $\bar{\tau} = \mu = 0.5$. When $k = 0.05$, Theorem 1 in [24] shows the system to be globally exponentially stable; but Theorem 1 in [15] fails to verify that. However, Theorem 1 in this paper in combination with the bisection search method yields $k = 0.67$, for which the system is globally exponentially stable.

In addition, Theorem 1 is also applicable to cases in which the derivative of the time delay is greater than or equal to 1. For example, if we let $\tau(t) = \sin^2 t$, then $\bar{\tau} = \mu = 1$. The bisection search method shows the system to be globally exponentially stable for $k = 0.54$. And we obtained an exponential convergence rate of $k = 0.37$ for $\tau(t) = 2\sin^2 t$. In contrast, the method in [15], [24] fails to verify exponential stability in either of these cases.

V. Conclusion

In this paper, free-weighting matrices are employed to express the relationship between the terms in the Leibniz-Newton formula, and an LMI-based delay-dependent exponential-stability criterion is derived for delayed neural networks with a time-varying delay. Two numerical examples demonstrate that this method is an improvement over existing ones.

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