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An Improved Global Asymptotic Stability Criterion for Delayed Cellular Neural Networks

Yong He, Min Wu, and Jin-Hua She, *Member, IEEE,*

***Abstract*— A new Lyapunov-Krasovskii functional is constructed for delayed cellular neural networks, and the S-procedure is employed to handle the nonlinearities. An improved global asymptotic stability criterion is also derived that is a generalization of, and an improvement over, previous results. Numerical examples demonstrate the effectiveness of the criterion.**

***Index Terms*— global asymptotic stability, delayed cellular neural networks, linear matrix inequality (LMI), S-procedure.**

I. Introduction

Cellular neural networks (CNNs) were first proposed in [1]. They have found application in many areas such as signal processing, pattern recognition, and static image processing. A CNN with a delay, which is called a delayed cellular neural network (DCNN), was first reported in [2] and has been the subject of many studies over the past few years (e.g., [3]–[21]). Among them, [3] and [4] presented some exponential stability criteria for DCNNs; but their treatment of nonlinearities by using inequalities was conservative. On the other hand, linear matrix inequalities (LMIs) are an efficient method of solving standard convex optimization problems numerically. Singh derived an LMI-based criterion [5], which was a generalization of, and an improvement over, previous criteria, such as [6]–[9]. Later, Zhang *et al.* extended those results to handle DCNNs with a time-varying delay and time-varying structured uncertainties [10]. However, there is room for further investigation. First, the constraints on the nonlinearities in [5] are very strict, i.e., the upper bounds of the sectors are all set to 1. Even though they were relaxed to k in [10], all the bounds were set to the same value. Second, the S-procedure [22], [23] was employed to deal with the constraints in [5]. However, the parameters in the S-procedure were exactly the same as those in the Lyapunov-Krasovskii functional, which may also lead to conservatism. Third, the Lyapunov-Krasovskii functional did not contain an integral term of state, which has proven to be very effective in handling delay systems.

This note presents a new Lyapunov-Krasovskii functional containing an integral term of state for DCNNs; and shows how the S-procedure can be employed to derive an LMI-based global asymptotic stability criterion. Moreover, it shows that the criterion in [5] is a special case of the one in this paper. Finally, numerical examples demonstrate the effectiveness of this approach, and that it is an improvement over previous ones.

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Y. He and M. Wu are with the School of Information Science and Engineering, Central South University, Changsha 410083, China. Corresponding author: Y. He, Email: heyong08@yahoo.com.cn

J.-H. She is with the School of Bionics, Tokyo University of Technology, Tokyo, 192-0982 Japan.

II. System Description

Consider the following DCNN:

$$\dot{x}(t) = -x(t) + Ag(x(t)) + A_\tau g(x(t-\tau(t))) + u, \qquad (1)$$

where $x(\cdot) = [x_1(\cdot), x_2(\cdot), \cdots, x_n(\cdot)]^T$ is the neuron state vector, $g(x(\cdot)) = [g_1(x_1(\cdot)), g_2(x_2(\cdot)), \cdots, g_n(x_n(\cdot))]^T$ is the output vector, and $u = [u_1, u_2, \cdots, u_n]^T$ is a constant input vector. A is a feedback matrix, and A_τ is a delayed feedback matrix. The delay, $\tau(t)$, is a time-varying differentiable function satisfying

$$\dot{\tau}(t) \le \mu, \qquad (2)$$

where μ is a constant. In addition, it is assumed that each neuron activation function in system (1), $g_j(\cdot),\ j = 1, 2, \cdots, n$, satisfies the following condition:

$$0 \le \frac{g_j(x) - g_j(y)}{x - y} \le k_j, \quad \forall\, x, y \in \mathcal{R},\ x \ne y,\ j = 1, 2, \cdots, n, \qquad (3)$$

where $k_j,\ j = 1, 2, \cdots, n$ are positive constants.

Now, we shift the equilibrium point $x^* = [x_1^*, x_2^*, \cdots, x_n^*]^T$ of system (1) to the origin by introducing a new state $z(\cdot) = x(\cdot) - x^*$, which transforms the system into the following:

$$\dot{z}(t) = -z(t) + A\phi(z(t)) + A_\tau \phi(z(t-\tau(t))), \qquad (4)$$

where $z(\cdot) = [z_1(\cdot), z_2(\cdot), \cdots, z_n(\cdot)]^T$ is the state vector of the transformed system, and $\phi(z(\cdot)) = [\phi_1(z_1(\cdot)), \phi_2(z_2(\cdot)), \cdots, \phi_n(z_n(\cdot))]^T$ and $\phi_j(z_j(\cdot)) = g_j(z_j(\cdot) + x_j^*) - g_j(x_j^*),\ j = 1, 2, \cdots, n$. Note that the functions $\phi_j(\cdot),\ j = 1, 2, \cdots, n$ satisfy the following conditions:

$$0 \le \frac{\phi_j(z_j)}{z_j} \le k_j, \quad \phi_j(0) = 0, \quad \forall z_j \ne 0,\ j = 1, 2, \cdots, n, \qquad (5)$$

which are equivalent to

$$\phi_j(z_j)\left[\phi_j(z_j) - k_j z_j\right] \le 0, \quad \phi_j(0) = 0, \quad j = 1, 2, \cdots, n. \qquad (6)$$

The S-procedure is employed to investigate the asymptotic stability of system (4), and is stated as follows:

Lemma 1: [22], [23] (S-procedure) Let $T_i \in R^{n\times n}$ $(i = 0, 1, \cdots, p)$ be symmetric matrices. The conditions on T_i $(i = 0, 1, \cdots, p)$,

$$\zeta^T T_0 \zeta > 0, \ \forall \zeta \neq 0 \text{ s.t. } \zeta^T T_i \zeta \geq 0 \ (i = 1, 2, \cdots, p), \tag{7}$$

hold if there exist $\tau_i \geq 0$ $(i = 1, 2, \cdots, p)$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0. \tag{8}$$

III. Stability Criterion

In this section, a new Lyapunov-Krasovskii functional containing an integral term of state is constructed. The S-procedure is employed to deal with nonlinearities. And the following asymptotic stability criterion is obtained.

Theorem 1: The origin of system (4) subject to conditions (5) and (2) is globally asymptotically stable if there exist $P = P^T > 0$, $R = R^T > 0$, $Q = Q^T > 0$, $D = \text{diag}(d_1, d_2, \cdots, d_n) \geq 0$, $T = \text{diag}(t_1, t_2, \cdots, t_n) \geq 0$, and $S = \text{diag}(s_1, s_2, \cdots, s_n) \geq 0$ such that the following LMI is feasible:

$$\Phi = \begin{bmatrix} \Phi_{11} & 0 & \Phi_{13} & PA_\tau \\ 0 & -(1-\mu)R & 0 & KS \\ \Phi_{13}^T & 0 & \Phi_{33} & DA_\tau \\ {A_\tau}^T P & SK & {A_\tau}^T D & -(1-\mu)Q - 2S \end{bmatrix} < 0, \tag{9}$$

where

$$\Phi_{11} = -2P + R,$$

$$\Phi_{13} = PA - D + KT,$$

$$\Phi_{33} = DA + A^T D + Q - 2T,$$

$$K = \text{diag}\{k_1, k_2, \cdots, k_n\}.$$

Proof: Construct the following Lyapunov-Krasovskii functional:

$$V(z(t)) = z^T(t)Pz(t) + 2\sum_{j=1}^{n} d_j \int_0^{z_j} \phi_j(s)ds + \int_{t-\tau(t)}^{t} \left[z^T(s)Rz(s) + \phi^T(z(s))Qf(z(s))\right] ds, \tag{10}$$

where $P = P^T > 0$, $R = R^T > 0$, $Q = Q^T > 0$, $D = \text{diag}(d_1, d_2, \cdots, d_n) \geq 0$ are to be determined. Calculating the derivative of $V(z(t))$ along the solution of system (4) yields

$$\begin{aligned} &\dot{V}(z(t)) \\ &= 2z^T(t)P\dot{z}(t) + 2\sum_{j=1}^{n} d_j \phi_j(z_j(t))\dot{z}_j(t) \\ &\quad + \left[||z(t)||_R^2 - (1-\dot{\tau}(t))||z(t-\tau(t))||_R^2\right] \\ &\quad + \left[||\phi(z(t))||_Q^2 - (1-\dot{\tau}(t))||\phi(z(t-\tau(t))||_Q^2\right] \\ &\leq 2z^T(t)P\dot{z}(t) + 2\phi^T(z(t))D\dot{z}(t) \\ &\quad + \left[||z(t)||_R^2 - (1-\mu)||z(t-\tau(t))||_R^2\right] \\ &\quad + \left[||\phi(z(t))||_Q^2 - (1-\mu)||\phi(z(t-\tau(t))||_Q^2\right], \end{aligned} \tag{11}$$

where

$$||x(t)||_Q^2 := x^T(t)Qx(t). \tag{12}$$

It is clear from (6) that

$$\phi_j(z_j(t))\left[\phi_j(z_j(t)) - k_j z_j(t)\right] \leq 0, \ j = 1, 2, \cdots, n \tag{13}$$

and

$$\phi_j(z_j(t-\tau(t)))\left[\phi_j(z_j(t-\tau(t))) - k_j z_j(t-\tau(t))\right] \leq 0, \quad j = 1, 2, \cdots, n \tag{14}$$

hold. Thus, by applying the S-procedure, we find that system (4) is asymptotically stable if there exist $T = \text{diag}(t_1, t_2, \cdots, t_n) \geq 0$ and $S = \text{diag}(s_1, s_2, \cdots, s_n) \geq 0$ such that

$$\begin{aligned} &\dot{V}(z(t)) - 2\sum_{j=1}^{n} t_j \phi_j(z_j(t))\left[\phi_j(z_j(t)) - k_j z_j(t)\right] \\ &-2\sum_{j=1}^{n} s_j \phi_j(z_j(t-\tau(t)))\left[\phi_j(z_j(t-\tau(t))) - k_j z_j(t-\tau(t))\right] \\ &\leq \xi^T(t)\Phi\xi(t) \\ &< 0 \end{aligned} \tag{15}$$

for all $\xi(t) \neq 0$, where

$$\xi(t) = [z^T(t), z^T(t-\tau(t)), \phi^T(z(t)), \phi^T(z(t-\tau(t)))]^T.$$

This completes the proof. ■

Remark 1: Unlike the Lyapunov-Krasovskii functionals in [5] and [10], the one above contains an integral term of state, $\int_{t-\tau(t)}^{t} x^T(s)Rx(s)ds$ (10). The advantage of this is that the delay term can be reserved so that the S-procedure can be applied to both the state and delay terms. Singh also employed the S-procedure in deriving the criterion in [5]. However, the same parameter, D, was used in both the S-procedure and the Lyapunov matrix; and the S-procedure was applied only to the state. In contrast, in Theorem 1, the Lyapunov matrix, D, is different from the parameter matrices, T and S, used in the S-procedure. This treatment allows us to fully exploit the potential of the S-procedure, and further reduces the conservatism. As for the nonlinear constraints, the cases $k_1 = k_2 = \cdots = k_n = 1$ and $k_1 = k_2 = \cdots = k_n = k$ were considered in [5] and [10], respectively. However, the S-procedure in this note can handle more general nonlinear constraints (6). In fact, if we set $T = D$, $S = 0$ and $R = \varepsilon I$, where ε is a sufficiently small positive scalar, then Theorem 1 in this note yields Theorem 1 in [5]. So, the matrices T, S and R provide extra freedom in parameter selection, which can potentially yield less conservative results.

IV. Examples

The two examples in this section demonstrate the validity of the new criterion.

Example 1: Consider the second-order DCNN (4) with the following parameters

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \ A_\tau = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}.$$

When $\mu = 0$, Theorem 1 in [5] can handle only the case $k_1 = k_2 = 1$. In contrast, using LMI (9) in Theorem 1, we found that the system is asymptotically stable for $k_1 = k_2 = 1.55$.

Moreover, Theorem 1 can also handle a time-varying delay. Table I lists the upper bounds on k_2 for $k_1 = 1.2$ and various μ when $\tau(t)$ is time-varying and k_1 is fixed.

TABLE I

UPPER BOUNDS ON k_2 FOR $k_1 = 1.2$ AND VARIOUS μ.

μ	0.1	0.2	0.3	0.4	0.5
k_2	7.15	2.49	1.19	0.58	0.21

Example 2: Consider the second-order DCNN (4) with the following parameters

$$A = \begin{bmatrix} 0.5 & 0.5 \\ -1 & -0.5 \end{bmatrix}, \; A_\tau = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0 \end{bmatrix}.$$

When $\mu = 0$, Theorem 1 in [5] and Theorem 1 in [10], as well as Theorem 1 in [3] and Theorem 1 in [4] ($k = 0$) all fail when $k_1 = k_2 = 1$. However, Theorem 1 in this note shows that the system is asymptotically stable in this case. Furthermore, it also shows that the asymptotic stability is guaranteed, even when $k_1 = 1$ and $k_2 = 1.55$.

V. CONCLUSION

This note presents a new Lyapunov-Krasovskii functional containing an integral term of state for DCNNs. The S-procedure was employed to deal with nonlinearities, and a less conservative global asymptotic stability criterion was derived. Numerical examples demonstrated the effectiveness of this approach and that it is an improvement over previous treatments.

REFERENCES

[1] L. O. Chua and L. Yang, "Cellular neural networks: thoery," *IEEE Trans. Circuits Syst.*, vol. 35, pp. 1257-1272, Oct. 1988.

[2] T. Roska, T. Boros, P. Thiran, and L.O. Chua, "Detecting simple motion using cellular neural networks," in *Proc. IEEE Int. Workshop on Cellular Neural Networks and Their Applications,* pp. 127-138.

[3] X. Liao, G. Chen, and E. N. Sanchez, "Delay dependent exponential stability analysis of delayed neural networks: an LMI approach," *Neural Networks*, vol. 15, pp. 855-866, 2002.

[4] E. Yucel and S. Arik, "New exponential stability results for delayed neural networks with time varying delays," *Physica D: Nonlinear Phenomena,* vol. 191, pp. 314-322, 2004.

[5] V. Singh, "A generalized LMI-based approach to the global asymptotic stability of delayed cellular neural networks," *IEEE Trans. Neural Networks,* vol. 15, pp. 223-225, Jan. 2004.

[6] J. Cao and D. Zhou, "Stability analysis of delayed cellular neural networks," *Neural Networks*, vol. 11, pp. 1601-1605, 1998.

[7] T. L. Liao and F. C. Wang, "Global stability for cellular neural networks with time delay," *IEEE Trans. Neural Networks*, vol. 11, pp. 1481-1484, Nov. 2000.

[8] J. Cao, "A set of stability criteria for delayed cellular neural networks," *IEEE Trans. Circuits Syst. I,* vol. 48, pp. 494-498, Apr. 2001.

[9] S. Arik, "An analysis of global asymptotic stability of delayed cellular neural networks," *IEEE Trans. Neural Networks,* vol. 13, pp. 1239-1242, Sept. 2002.

[10] H. Zhang, C. Li, and X. Liao, "A note on the robust stability of neural networks with time delay," *Chaos, Solitons and Fractals*, vol. 25, pp. 357-360, 2005.

[11] T. Roska, C. W. Wu, and L.O. Chua, "Stability and dynamics of delay-type general and cellular neural networks," *IEEE Trans. Circuits Syst. I*, vol. 39, pp. 487-490, June 1992.

[12] J. Cao, "Global stability analysis in delayed cellular neural networks," *Physical Rev. E,* vol. 59, pp. 5940-5944, 1999.

[13] S. Arik and V. Tavsanoglu, "On the global asymptotic stability of delayed cellular neural networks," *IEEE Trans. Circuits Syst. I,* vol. 47, pp. 571-574, Apr. 2000.

[14] X. Liao, K. Wong, and J. Yu, "Novel stability conditions for cellular neural networks with time delay," *Int. J. Bifurc. Chaos*, vol. 11, pp. 1853-1864, 2001.

[15] J. Cao, "Global stability conditions for delayed CNNs," *IEEE Trans. Circuits Syst. I,* vol. 48, pp. 1330-1333, Nov. 2001.

[16] S. Arik, "An improved global stability result for delayed cellular neural networks," *IEEE Trans. Circuits Syst. I,* vol. 49, pp. 1211-1214, Aug. 2002.

[17] X. Liao, Z. Wu and J. Yu, "Stability analysis of cellular neural networks with continuous time delays," *J. Comput. Appl. Math.*, vol. 143, pp. 29-47, 2002.

[18] X. X. Liao and J. Wang, "Algebraic criteria for global exponential stability of cellular neural networks with multiple time delays," *IEEE Trans. Circuits Syst. I,* vol. 50, pp. 268-275, Feb. 2003.

[19] G. J. Yu, C. Y. Lu, J. S. H. Tsai, B. D. Liu, and T. J. Su, "Stability of cellular neural networks with time-varying delay," *IEEE Trans. Circuits Syst. I,* vol. 50, pp. 677-678, May 2003.

[20] S. Arik, "An analysis of exponential stability of delayed neural networks with time varying delays," *Neural Networks*, vol. 17, pp. 1027-1031, 2004.

[21] Q. Zhang, X. Wen, and J. Xu, "Delay-dependent exponential stability of cellular neural networks with time-varying delays," *Chaos, Solitons and Fractals,* vol. 23, pp. 1363-1369, 2005.

[22] V.A. Jakubovič, "The S-procedure in nonlinear control theory," *Vestnik Leningard Univ Math,* vol. 4, pp. 73-93, 1977.

[23] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory,* Philadelphia, PA: SIAM, 1994.