A New Approach to the Estimation and Rejection of Disturbances in Servo Systems

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Abstract— This paper proposes a new approach to disturbance estimation based on a curvature model to improve the disturbance rejection performance of a servo system. The main feature is that the stability of the control system is guaranteed when the disturbance estimate is incorporated directly into the designed servo control law. Experimental results show that disturbances are rejected efficiently when this approach is used.

Index Terms— servo system, circle of curvature, disturbance estimation, disturbance rejection, globally uniformly ultimately bounded (GUUB), optimal control.

I. INTRODUCTION

DISTURBANCE rejection is an important issue in the design of servo systems. It is well known that the perfect rejection of a disturbance in a servo system can be achieved either by using a feedforward control strategy if we have complete knowledge of the disturbance or can measure it directly, or by inserting an internal model of the disturbance generator into the servo controller if the key characteristics of the disturbance are known [1]. However, it is rare that we have complete knowledge of the disturbance or can make use of information about it directly. Usually, we do not even know all the characteristics of a disturbance because a disturbance in the system usually covers a wide frequency band in many control applications. So, it is difficult to provide the desired rejection performance. While several methods of rejecting disturbances have been proposed (e.g., [2]–[4]) to improve the performance, they require some *a priori* information about a disturbance; otherwise, the design of the controller is complicated.

In this paper, a new approach to disturbance estimation based on a curvature model is proposed to improve the performance of disturbance rejection in a servo system. The characteristics of this method are that disturbances are reproduced satisfactorily even though the estimation model is very simple; the stability of the system is guaranteed when the disturbance estimate is incorporated directly into the designed servo control law; and no *a priori* information about a disturbance, such as the peak value, is needed.

Throughout this paper, I_n means an $n \times n$ identity matrix; $A_{n \times m}$ indicates a matrix with n rows and m columns; $\|A\|$ is the Euclidean norm of matrix or vector A; and $\mathcal{O}_a^b(\tau^k)$ is an infinitesimal with the same order as τ^k. For a vector-valued sequence $x(k), k = 0, 1, \cdots, \|x\|_\infty = \sup_k \|x(k)\|$; and for a system G, $\|G\|_1 = \sup_{\|w\|_\infty = 1} \|Gw\|_\infty$.

This paper is organized as follows. The design of a servo controller is outlined in Section II. The method of disturbance estimation is described in Section III. Then, the incorporation of the estimate into a servo control law in order to reject disturbances is explained in Section IV. Section V gives some experimental results to show the validity of the method, and some concluding remarks are made in Section VI.

II. DESIGN OF SERVO CONTROLLER

The configuration of a conventional servo system is shown in Fig. 1. An exogenous disturbance, $d(k)$, is assumed to be added to the input channel. The plant, $P(z)$, and the servo controller, $K(z)$, are respectively given by

$$P : \begin{cases} x_P(k+1) = A_P x_P(k) + B_P[u(k) + d(k)], \\ y(k) = C_P x_P(k), \end{cases} \tag{1}$$

and

$$K : \begin{cases} x_K(k+1) = A_K x_K(k) + B_K e(k), \\ u(k) = C_K x_K(k) + C_{KP} x_P(k) + D_K e(k), \end{cases} \tag{2}$$

where $x_P(k) \in \mathcal{R}^{n_P}$, $x_K(k) \in \mathcal{R}^L$, $y(k) \in \mathcal{R}$, $u(k) \in \mathcal{R}$, $d(k) \in \mathcal{R}$ and $e(k) \in \mathcal{R}$ are the states of the plant and servo controller, output, control input, disturbances and tracking error, respectively. The following assumptions are made in this study.

Assumption 1: (A_P, B_P) is controllable.

Assumption 2: The state of the plant, $x_P(k)$, is available.

Assumption 3: The disturbance, $d(k)$, is bounded and smooth enough.

Many approaches to the design of a servo controller have been proposed (e.g., [1], [5]–[7]). In what follows, we show an optimal design method for a servo controller. $d(k) = 0$ is assumed in order to focus on the tracking problem.

Let the reference input be generated by

$$\phi(z^{-1}) = 1 + \phi_{L-1} z^{-1} + \cdots + \phi_0 z^{-L}, \tag{3}$$

i.e.,
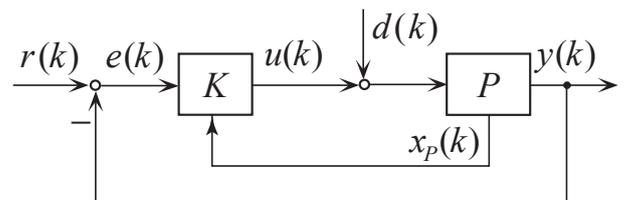
$$\phi(z^{-1}) r(k) = 0.$$

Fig. 1. Conventional servo system.

Then, A_K and B_K in (2) can be written as

$$A_K = \begin{bmatrix} 0_{(L-1)\times 1} & I_{L-1} \\ -\phi_0 & \{-\phi_1 \cdots -\phi_{L-1}\} \end{bmatrix},$$

$$B_K = \begin{bmatrix} 0_{(L-1)\times 1} \\ 1 \end{bmatrix}.$$

Note that

$$\phi(z^{-1})e(k+1) = \phi(z^{-1})r(k+1) - \phi(z^{-1})y(k+1)$$
$$= -\phi(z^{-1})C_P \{A_P x_P(k) + B_P u(k)\},$$

and let $\Delta x_P(k) = \phi(z^{-1})x_P(k)$ and $\Delta u(k) = \phi(z^{-1})u(k)$. The dynamics of the controller (2) can then be rewritten in terms of the tracking error as follows:

$$\epsilon(k+1) = A_K \epsilon(k) + A_{KP}\Delta x_P(k) + B_{KP}\Delta u(k), \qquad (4)$$

$$\epsilon(k) := [e(k-L+1)\ e(k-L+2)\ \cdots\ e(k)]^T,$$

$$A_{KP} = \begin{bmatrix} 0_{(L-1)\times n_P} \\ -C_P A_P \end{bmatrix}, \quad B_{KP} = \begin{bmatrix} 0_{(L-1)\times 1} \\ -C_P B_P \end{bmatrix}.$$

The relationship between $e(k-i)$ and $x_{K(L-i)}(k)$, a component of $x_K(k) = [x_{K1}(k)\ \cdots\ x_{KL}(k)]^T$, is

$$e(k-i) = \phi(z^{-1})x_{K(L-i)}(k+1),\ i = 0, \cdots, L-1. \quad (5)$$

Multiplying both sides of the system equation in (1) by $\phi(z^{-1})$ and combining it with (4) yields the following single augmented state representation of the whole system.

$$\xi(k+1) = A\xi(k) + B\Delta u(k), \qquad (6)$$

$$\xi(k) := \begin{bmatrix} \epsilon^T(k) & \Delta x_P^T(k) \end{bmatrix}^T,$$

$$A = \begin{bmatrix} A_K & A_{KP} \\ 0_{n_P \times L} & A_P \end{bmatrix}, \quad B = \begin{bmatrix} B_{KP} \\ B_P \end{bmatrix}.$$

Now, the design problem can be stated as:

Design a state feedback controller

$$\Delta u(k) = F_G \xi(k) = [f_K\ f_P][\epsilon^T(k)\ \Delta x_P^T(k)]^T \qquad (7)$$

that guarantees the internal stability of the servo system.

An optimal controller is designed by minimizing the performance index

$$J = \sum_{k=0}^{\infty}[\xi^T(k)Q\xi(k) + \rho\Delta u^2(k)], \qquad (8)$$
$$Q \geq 0, \quad \rho > 0,$$

and the resulting control law is given by

$$P = Q + A^T P A - A^T P B(\rho + B^T P B)^{-1} B^T P A,$$
$$F_G = -(\rho + B^T P B)^{-1} B^T P A.$$

Substituting (5) into (7) and dividing (7) by $\phi(z^{-1})$ yields the control law

$$u(k) = [f_K\ f_P][x_K^T(k+1)\ x_P^T(k)]^T. \qquad (9)$$

Furthermore, if we let

$$f_K = \begin{bmatrix} f_0 & f_1 & \cdots & f_{L-1} \end{bmatrix},$$

the feedback gains in (2) are

$$C_K = [-f_{L-1}\phi_0\ f_0 - f_{L-1}\phi_1\ \cdots\ f_{L-2} - f_{L-1}\phi_{L-1}],$$
$$C_{KP} = f_P, \quad D_K = f_{L-1}.$$

Remark 1: In the design of the servo controller, $K(z)$, we assumed that $d(k) = 0$. It is well known that disturbance rejection performance depends on the sensitivity function, S, of the system; and optimal control gives $|S| \leq 1$ [7]. So, generally speaking, the designed servo controller suppresses disturbances as well. And the disturbance rejection performance can further be tuned by choosing suitable diagonal elements related to $\epsilon(k)$ in the weighting matrix, Q, in (8), which is associated with S.

III. DISTURBANCE ESTIMATION

Perfect disturbance rejection is obtained for signals for which the controller, $K(z)$, contains an internal model. However, if the controller does not contain an internal model of the disturbance, good rejection performance cannot be expected. Generally speaking, the peak value of the tracking error is proportional to the peak value of the disturbance. If some *a priori* information about a disturbance, e.g. the peak value, is known, a nonlinear control law can be designed to reject it [8]. In this paper, we do not use such *a priori* information. The only assumption about a disturbance is that the sampling frequency is high enough that the disturbance is smooth enough.

Haskara *et al.* [2] have proposed a method of estimating disturbances using a linear model. In their method, the order of the estimator must be very high in order to obtain a precise estimate. In contrast, the estimation model described in this paper is of a low order; and in spite of that, the estimates are very precise. More specifically, the precision is proportional to the square of the sampling time.

Komada and Ohnisih [3] have proposed a method called disturbance observer to estimate a disturbance, and the method has been applied to several electro-mechanical systems [4], [9], [10]. In their method, the disturbance is first described by

$$d(k) = \frac{1}{P}y(k) - u(k). \qquad (10)$$

Since $1/P(z)$ is not proper, the disturbance cannot be obtained directly from (10). A low-pass filter, $F(z)$, is used to make $F(z)/P(z)$ proper, and the disturbance is estimated by

$$\hat{d}(k) = \frac{F}{P}y(k) - u(k). \qquad (11)$$

Note that (11) cannot be used for a continuous plant with unstable poles/zeros because unstable pole-zero cancellations would occur. Even if a continuous plant has no unstable poles/zeros, (11) still cannot be used when the relative degree of the continuous plant is higher than two because unstable limiting zeros occur in the pulse-transfer function of the plant. So, special techniques are required to use a discrete-time disturbance observer to estimate disturbances. Furthermore, since the stability of the system is not guaranteed when the disturbance estimate is incorporated directly into the designed control law, the issue of the stability of the whole system must be taken into account in the design of the low-pass filter, $F(z)$. So, the construction of $F(z)$ may be complicated. In contrast, one feature of the method described in this paper is that the stability of the whole system is guaranteed when the estimate is incorporated directly into the designed control law.
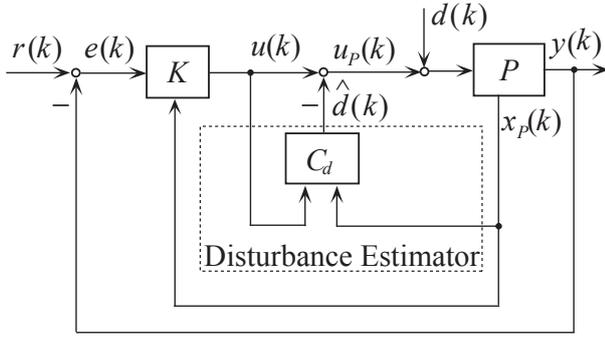
Fig. 2.   Configuration of proposed servo system.

Fig. 3.   Circle-of-curvature model for disturbance estimation.

In this paper, a low-order nonlinear disturbance-estimation model called a curvature model is used to estimate disturbances and reduce the tracking error. The configuration of the proposed servo system is shown in Fig. 2. It results from plugging a nonlinear disturbance estimator, C_d, into a conventional servo system, and has a structure similar to that of a two-degree-of-freedom servo system [11]. So roughly speaking, the rejection of disturbances is mainly handled by the controller C_d, and the reference tracking is primarily handled by the controller $K(z)$.

A circle of curvature approximation approximates the curve around the point $(k-1)\tau$ using an arc of the circle of curvature at $(k-1)\tau$. Here, this method is employed to estimate a disturbance. If the circle of curvature at $(k-1)\tau$ is known, then the value on this circle at $k\tau$ can be considered to be an estimate of the disturbance at $k\tau$ (see Fig. 3). This estimate has the following characteristics:

1) The circle of curvature shares the same tangent line with the disturbance at $(k-1)\tau$.
2) The circle of curvature has the same concavity or convexity as the disturbance at $(k-1)\tau$.
3) The curvature of the circle of curvature equals that of the disturbance at $(k-1)\tau$.

So, the characteristics of the disturbance are reflected in the estimate; and by utilizing the estimate, the disturbance can effectively be suppressed. The details are given below.

According to Assumption 1, there exists a nonsingular matrix $T \in \mathcal{R}^{n_P \times n_P}$ that converts the plant (1) into the following controllability canonical form:

$$\begin{cases} \bar{x}_P(k+1) = \bar{A}_P \bar{x}_P(k) + \bar{B}_P[u(k) + d(k)], \\ y(k) = \bar{C}_P \bar{x}_P(k), \end{cases} \quad (12)$$

where

$$\bar{A}_P = T^{-1} A_P T = \begin{bmatrix} 0_{(n_P-1) \times 1} & I_{n_P - 1} \\ -\alpha_1 & \{-\alpha_2 \ \cdots \ -\alpha_{n_P}\} \end{bmatrix},$$

$$\bar{B}_P = T^{-1} B_P = \begin{bmatrix} 0_{1 \times (n_P - 1)} & 1 \end{bmatrix}^T, \quad \bar{C}_P = C_P T.$$

Multiplying both sides of (12) by $\bar{B}_P^T$ gives

$$\bar{B}_P^T \bar{x}_P(k+1) = D \bar{x}_P(k) + u(k) + d(k),$$
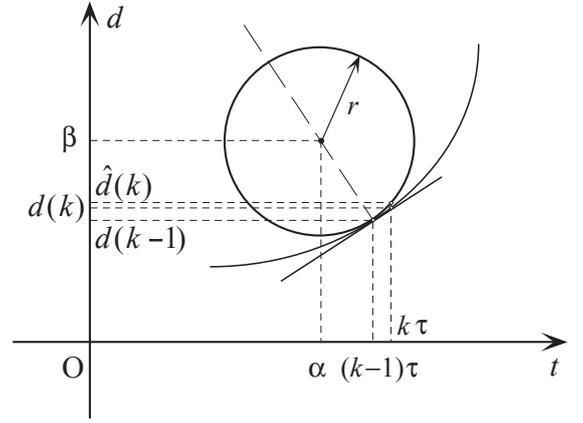$$D := \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n_P} \end{bmatrix}.$$

So, the disturbance, $d(k)$, can be expressed as

$$\begin{aligned} d(k) &= \bar{B}_P^T \bar{x}_P(k+1) - D\bar{x}_P(k) - u(k) \\ &= \hat{B}_P^T x_P(k+1) - \hat{D} x_P(k) - u(k), \quad (13) \end{aligned}$$

where

$$\hat{B}_P^T = B_P^T T^{-T} T^{-1}, \quad \hat{D} = D T^{-1}.$$

As a result, the disturbance up to $(k-1)\tau$ can be calculated using the above equation, and the following equations hold:

$$\begin{cases} d(k-1) = \hat{B}_P^T x_P(k) - \hat{D} x_P(k-1) - u(k-1), \\ d(k-2) = \hat{B}_P^T x_P(k-1) - \hat{D} x_P(k-2) - u(k-2), \\ d(k-3) = \hat{B}_P^T x_P(k-2) - \hat{D} x_P(k-3) - u(k-3). \end{cases}$$
$$(14)$$

For a sampling period, τ, if the first and second derivatives of $d(k)$ at $(k-1)\tau$ are approximated by

$$\hat{d}'(k-1) \approx \frac{d(k-1) - d(k-2)}{\tau},$$

$$\hat{d}''(k-1) \approx \frac{d(k-1) - 2d(k-2) + d(k-3)}{\tau^2},$$

then the radius, r, of the circle of curvature is

$$r^2 = \frac{\left[1 + \hat{d}'(k-1)^2\right]^3}{\hat{d}''(k-1)^2}, \quad (15)$$

and the coordinates of the center are

$$\begin{cases} \alpha = (k-1)\tau - \dfrac{\hat{d}'(k-1)\left[1 + \hat{d}'(k-1)^2\right]}{\hat{d}''(k-1)}, \\ \beta = d(k-1) + \dfrac{1 + \hat{d}'(k-1)^2}{\hat{d}''(k-1)}. \end{cases} \quad (16)$$

Thus, the disturbance estimate, $\hat{d}(k)$, is obtained from the following lemma.

Lemma 1: The disturbance estimate, $\hat{d}(k)$, is given by

$$\hat{d}(k) = \begin{cases} \beta - \sqrt{r^2 - (k\tau - \alpha)^2}, & \hat{d}''(k-1) > 0, \\ d(k-1) + \tau \hat{d}'(k-1), & \hat{d}''(k-1) = 0, \\ \beta + \sqrt{r^2 - (k\tau - \alpha)^2}, & \hat{d}''(k-1) < 0, \end{cases} \quad (17)$$

where r, α and β are given by (15) and (16).

IV. DISTURBANCE REJECTION

Combining the designed servo control law (9) with the disturbance estimate (17) yields the control law

$$u_P(k) = u(k) - \hat{d}(k). \tag{18}$$

The following theorem holds for this law.

Theorem 1: The control law (18) guarantees the stability of the control system and suppresses disturbances when the sampling period, τ, is small enough.

In order to prove the above theorem, we first give the discrete time version of the concept of globally uniformly ultimately bounded (GUUB) [12], and then show that the proposed control system is stable in the sense of GUUB.

Definition 1: The solution $x(k)$ of the system $x(k+1) = f(x(k), k)$ is said to be GUUB if there exists a positive constant κ_0 for a given positive constant b such that

$$\|x(k)\| \le b, \quad \forall k \ge \kappa_0 + \kappa$$

is satisfied regardless of the initial state, $x(k_0)$.

Proof: Assume that the internal model contained in $K(z)$ is $1/\phi(z^{-1})$, where $\phi(z^{-1})$ is defined in (3). According to Assumption 3, there exists a positive number, d_{eM}, such that

$$\|\phi(z^{-1})d(k)\|_\infty = d_{eM} < \infty. \tag{19}$$

Since the designed servo system without disturbance estimation is stable, there exists a positive number $\mathcal{K} < \infty$ such that

$$\left\| \begin{matrix} \phi(z^{-1})x_K(k) \\ \phi(z^{-1})x_P(k) \end{matrix} \right\|_\infty = \mathcal{K}\|\phi(z^{-1})d(k)\|_\infty = \mathcal{K}d_{eM}. \tag{20}$$

On the other hand, the Taylor expansion of $d(k-2)$ at $(k-1)\tau$ is

$$d(k-2) = d(k-1) - d'(k-1)\tau + \mathcal{O}_1^1(\tau^2),$$

or equivalently

$$d'(k-1) = \hat{\tilde{d}}(k-1) + \mathcal{O}_1^1(\tau). \tag{21}$$

In the same manner,

$$d'(k-2) = \hat{\tilde{d}}(k-2) + \mathcal{O}_1^2(\tau). \tag{22}$$

And the Taylor expansion of $d'(k-2)$ at $(k-1)\tau$,

$$d'(k-2) = d'(k-1) - d''(k-1)\tau + \mathcal{O}_2(\tau^2),$$

gives

$$d''(k-1) = \hat{\tilde{d}}''(k-1) + \mathcal{O}_2(\tau). \tag{23}$$

When $\hat{\tilde{d}}''(k-1) > 0$, the disturbance estimate is

$$
\begin{aligned}
\hat{d}(k) &= \beta - \sqrt{r^2 - (k\tau - \alpha)^2} \\
&= d(k-1) + \frac{1 + \hat{\tilde{d}}'(k-1)^2}{\hat{\tilde{d}}''(k-1)} \\
&\quad - \sqrt{\left[\frac{[1+\hat{\tilde{d}}'(k-1)^2]^3}{\hat{\tilde{d}}''(k-1)^2} - \left\{ \tau + \frac{\hat{\tilde{d}}'(k-1)[1+\hat{\tilde{d}}'(k-1)^2]}{\hat{\tilde{d}}''(k-1)} \right\}^2 \right]} \\
&= d(k-1) + \frac{1 + \hat{\tilde{d}}'(k-1)^2}{\hat{\tilde{d}}''(k-1)} \times \\
&\quad \left(1 - \sqrt{1 - \left\{ \frac{2\hat{\tilde{d}}'(k-1)\hat{\tilde{d}}''(k-1)}{1+\hat{\tilde{d}}'(k-1)^2} + \frac{\hat{\tilde{d}}''(k-1)^2}{[1+\hat{\tilde{d}}'(k-1)^2]^2}\tau \right\}\tau} \right) \\
&= d(k-1) + \hat{\tilde{d}}'(k-1)\tau + \tfrac{1}{2}\hat{\tilde{d}}''(k-1)\tau^2 + \mathcal{O}_{est}(\tau^3),
\end{aligned}
$$

where the following relationship is used in the derivation:

$$\sqrt{1-\chi} = 1 - \frac{1}{2}\chi - \frac{1}{2\cdot 4}\chi^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}\chi^3 - \cdots, \quad |\chi| \le 1.$$

The condition $|\chi| \le 1$ is guaranteed for a small τ. Since the Taylor expansion of $d(k)$ at $(k-1)\tau$ is

$$d(k) = d(k-1) + d'(k-1)\tau + \frac{1}{2}d''(k-1)\tau^2 + \mathcal{O}_0(\tau^3),$$

then

$$
\begin{aligned}
\Delta d(k) &:= d(k) - \hat{d}(k) \\
&= \left\{ d'(k-1) - \hat{\tilde{d}}'(k-1) \right\}\tau \\
&\quad + \frac{1}{2}\left\{ d''(k-1) - \hat{\tilde{d}}''(k-1) \right\}\tau^2 + \mathcal{O}_{err}(\tau^3).
\end{aligned}
$$

From (21) and (23) we obtain

$$\|\Delta d(k)\|_\infty = \mathcal{O}(\tau^2). \tag{24}$$

The above equation also holds for $\hat{\tilde{d}}''(k-1) < 0$ and $\hat{\tilde{d}}''(k-1) = 0$. So, if a small enough τ is chosen, then $\Delta d(k)$ will be bounded. In general, if the effects of a disturbance cannot be ignored, then $\|d(k)\|_\infty >> \mathcal{O}(\tau^2)$. Therefore,

$$\|\Delta d(k)\|_\infty << \|d(k)\|_\infty \tag{25}$$

is satisfied, and

$$
\begin{aligned}
\|\phi(z^{-1})\Delta d(k)\|_\infty &= \|\phi(z^{-1})\|_1 \|\Delta d(k)\|_\infty \\
&<< \|\phi(z^{-1})\|_1 \|d(k)\|_\infty = \|\phi(z^{-1})d(k)\|_\infty
\end{aligned}
$$

holds. The above yields

$$\|\phi(z^{-1})\Delta d(k)\|_\infty << d_{eM}. \tag{26}$$

So, in the improved servo system in Fig. 2, the equivalent disturbance added to the plant is $\Delta d(k)$, which is much smaller than the actual disturbance $d(k)$. If we incorporate the estimated disturbance into the servo control law, the following holds:

$$\left\| \begin{matrix} \phi(z^{-1})x_K(k) \\ \phi(z^{-1})x_P(k) \end{matrix} \right\|_\infty = \mathcal{K}\|\phi(z^{-1})\Delta d(k)\|_\infty << \mathcal{K}d_{eM}. \tag{27}$$

So,

$$
\begin{aligned}
\left\| \begin{matrix} \phi(z^{-1})x_K(k) \\ \phi(z^{-1})x_P(k) \end{matrix} \right\| &\le \sqrt{L+n_P} \left\| \begin{matrix} \phi(z^{-1})x_K(k) \\ \phi(z^{-1})x_P(k) \end{matrix} \right\|_\infty \\
&<< \sqrt{L+n_P}\,\mathcal{K}d_{eM}
\end{aligned} \tag{28}
$$

holds for all $k \ge 0$. It means that the control system is GUUB, and thus stable; and the effects of a disturbance are suppressed when the estimated disturbance is combined with the designed servo control law. ∎

Remark 2: The above theorem shows that, if the original servo system is stable, the system is still stable after directly plugging in the nonlinear disturbance estimator, C_d. This result can be viewed as a robustness property of the servo system, i.e., the system is robust with regard to the incorporation of the disturbance estimate.

Remark 3: If the servo control law $u_P(k)$, (18), which incorporates the disturbance estimate, is applied to the plant, the control input, $u(k)$, used in the calculation of disturbances (14) should be replaced by $u_P(k)$.
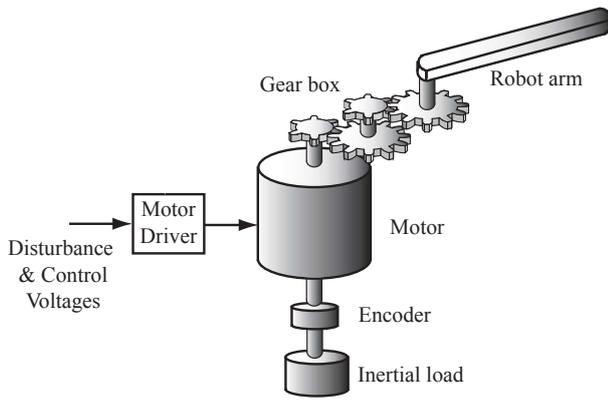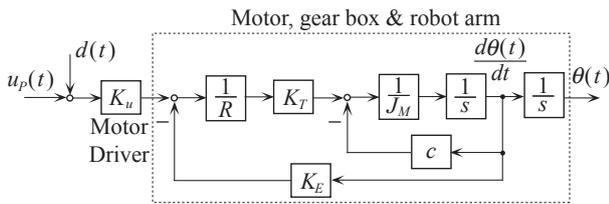
Fig. 4.   Arm robot.

Fig. 5.   Block diagram of arm robot.

Fig. 6.   Experimental system.

Fig. 7.   Block diagram of interface box.

V. Experiments

We applied the proposed method to the positioning control of an arm robot (Fig. 4) to verify its validity. The arm was driven by a Mabuchi DC motor (rated voltage: 3 V; rated current: 0.56 A; rated speed: 895 rad/s). The block diagram of the arm robot is shown in Fig. 5, where K_u is the voltage gain of the motor driver; R $[\Omega]$ is the resistance of the armature; J_M $[\mathrm{kg\,m^2}]$ is the moment of inertia of the motor, the gear box and the arm; c $[\mathrm{Nm\,s/rad}]$ is the frictional damping constant of the system; and K_T $[\mathrm{Nm/A}]$ and K_E $[\mathrm{Vs/rad}]$ are the torque constant and the back electromotive force constant, respectively. The output of the plant is the rotational angle $\theta(t)$ $[\mathrm{rad}]$, and the inputs are the control voltage $u_P(t)$ $[\mathrm{V}]$ and the disturbance voltage $d(t)$ $[\mathrm{V}]$.

The plant in the continuous time domain is

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} [u_P(t) + d(t)], \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \\
x(t) &:= \begin{bmatrix} \theta(t) & d\theta(t)/dt \end{bmatrix}^T, \\
a &= \frac{K_T K_E + cR}{J_M R} = 6.43 \ \text{s/rad}, \\
b &= \frac{K_T K_u}{J_M R} = 39.03 \ \text{V}^{-1}.
\end{aligned}
$$

The rotational angle $\theta(t)$ is measured with an optical encoder, and the rotational speed $d\theta(t)/dt$ is obtained by performing a digital differential operation on the rotational angle. The sampling period

$$
\tau = 0.01 \ \text{s} \tag{29}
$$

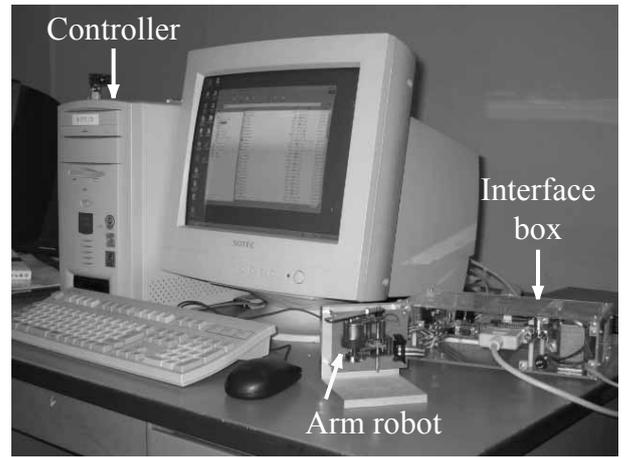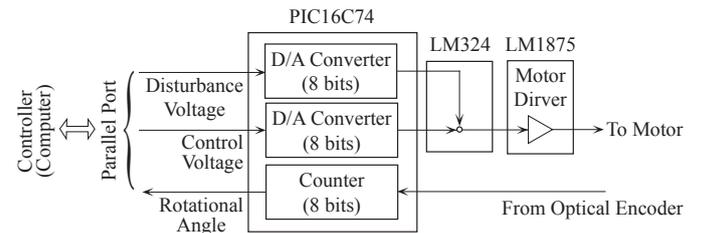is used to discretize the continuous plant; and the plant in the discrete time domain is

$$
A_P = \begin{bmatrix} 1 & 9.68528 \times 10^{-3} \\ 0 & 9.37724 \times 10^{-1} \end{bmatrix}, \quad B_P = \begin{bmatrix} 1.91010 \times 10^{-3} \\ 3.78017 \times 10^{-1} \end{bmatrix},
$$
$$
C_P = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{30}
$$

A photograph of the experimental system is shown in Fig. 6. A desktop computer (400-MHz Celeron) was used for control. A motor driver, a counter, and two D/A converters were built into the interface box, as shown in Fig. 7. A parallel connection was used between the interface box and the computer. The rotational speed was reduced by a gear box (64.8:1), and an optical encoder (16 cycles per turn) was mounted on the shaft of the motor to measure the angle of the arm. So, the resolution of the arm movement is 6.060×10^{-3} rad/pulse. Pulses from the encoder were sent to the counter in the interface box. The control input was fed to the motor through the interface box.

The reference input

$$
r(k) = \sin \frac{\pi}{100} k \tag{31}
$$

is added. The internal model of the reference input is given by

$$
\phi(z^{-1}) = 1 + \phi_1 z^{-1} + z^{-2}, \quad \phi_1 = -1.99901; \tag{32}
$$

and the matrices in the system equation of the servo controller are

$$
A_K = \begin{bmatrix} 0 & 1 \\ -1 & 1.99901 \end{bmatrix}, \quad B_K = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{33}
$$

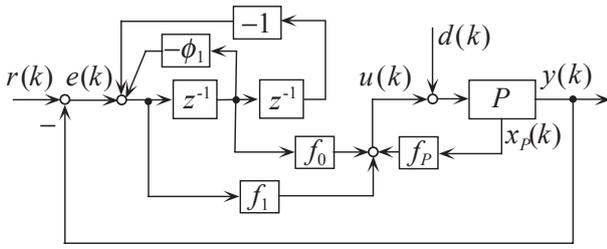First, choosing

$$
Q = I_4, \quad \rho = 10000 \tag{34}
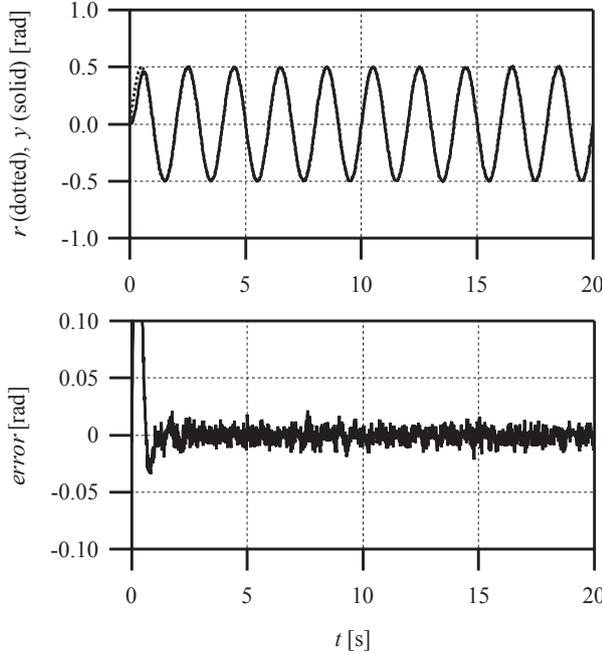$$

Fig. 8.   Optimal tracking control system.

Fig. 9.   Response of optimal tracking control system without disturbance.

Fig. 10.   Disturbance.

and optimizing the following performance index

$$J := \sum_{k=0}^{\infty} \left[\xi^T(k) Q \xi(k) + \rho \Delta u(k)^2 \right],$$
$$\xi(k) = \left[e(k-1) \quad e(k) \quad \phi(z^{-1}) x_P^T(k) \right]^T,$$
$$\Delta u(k) = \phi(z^{-1}) u(k)$$

yields the optimal servo control law

$$\begin{aligned}
u(k) &= \left[f_0 \quad f_1 \right] x_K(k+1) + f_P x_P(k), \\
&= \left[-2.20533 \quad 2.25216 \right] x_K(k+1) \\
&\quad + \left[-37.8972 \quad -2.66674 \right] x_P(k).
\end{aligned} \tag{35}$$

The optimal servo control system is shown in Fig. 8.

The tracking control results when no disturbance was input are shown in Fig. 9. The steady state tracking error is in the range ± 0.021 rad.

Next, the disturbance

$$\begin{aligned}
d(k) = \; & -2.5 \cos \frac{3\pi}{200} k - 2 \sin \frac{3\pi}{400} k - 1.5 \cos \frac{3\pi}{800} k \\
& - \sin \frac{3\pi}{1200} k - 0.5 \cos \frac{3\pi}{1600} k \; [\text{V}]
\end{aligned} \tag{36}$$

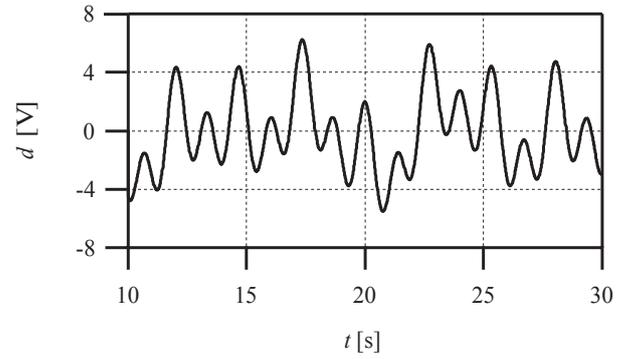was input (Fig. 10). The experimental results for the optimal system are shown in Fig. 11. Since no internal model of the

disturbance is contained in the servo controller, the disturbance cannot be rejected completely. In the steady state, the tracking error increases to ± 0.072 rad. The peak value of the power spectral density of the tracking error is 0.570, which appears at 4.86 rad/s, the angular frequency of the largest component of the disturbance. Next, the disturbance was estimated using the method proposed in this paper. The disturbance and the corresponding estimate are shown in Fig. 12. It is clear from the figure that the estimate reproduces the disturbance satisfactorily. The experimental results for a control law that makes use of the estimate are shown in Fig. 13. It can be seen that the system remains stable, and the steady-state tracking error drops to ± 0.025 rad. The power spectral density of the tracking error shows that the disturbance is almost completely rejected, except at an angular frequency of 4.86 rad/s; and even at that angular frequency, the peak value drops to 0.0355, which is less than one-sixteenth of that without the estimate. A comparison of Figs. 11 and 13 reveals that making use of the estimated disturbance significantly reduces the tracking error.

VI. CONCLUSIONS

To improve the disturbance rejection performance of a servo system, this paper proposes a curvature model for disturbance estimation, and an improved servo control law that makes use of the estimate. Unlike other approaches, we do not assume that any information about disturbances, such as the peak value, is known. The main features of this method are:

1) Disturbances are reproduced satisfactorily even though the estimation model is very simple.
2) The stability of the servo system is guaranteed when the disturbance estimate is incorporated directly into the designed servo control law, i.e., the system is robust.

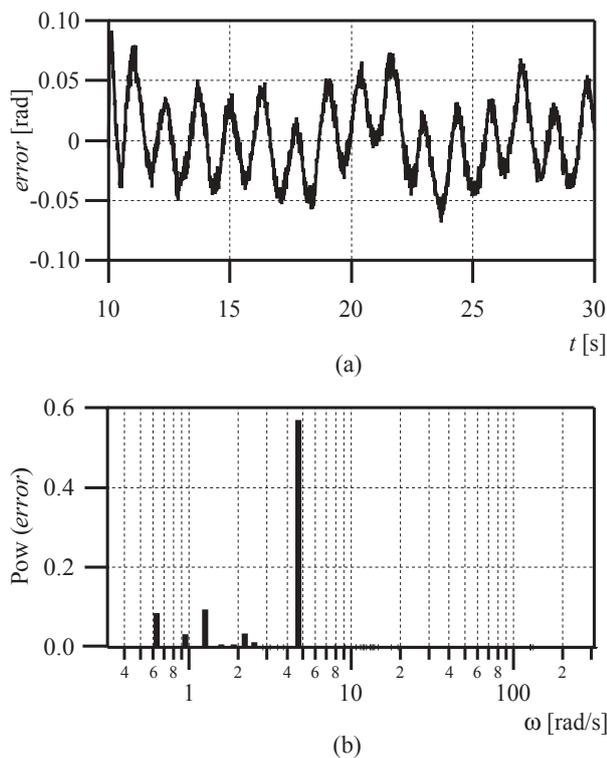The validity of the proposed method has been demonstrated through experiments.

Fig. 11. Response of optimal tracking control system without disturbance estimation. (a) Time trace. (b) Power spectral density.
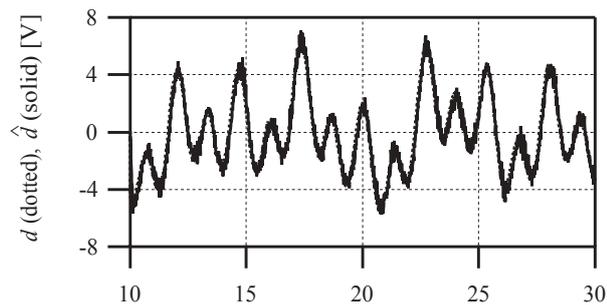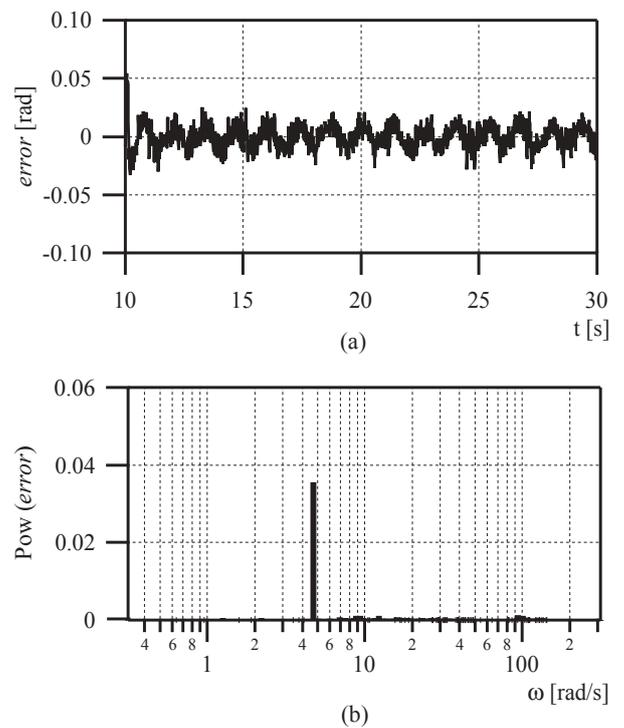

Fig. 12. Disturbance estimate.

Fig. 13. Response of optimal servo control system with disturbance estimation. (a) Time trace. (b) Power spectral density.

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