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Parameter-Dependent Lyapunov Functional for Stability of Time-Delay Systems with Polytopic-Type Uncertainties

Yong He, Min Wu, Jin-Hua She, *Member, IEEE,* and Guo-Ping Liu, *Member, IEEE,*

***Abstract*— This paper concerns the problem of the robust stability of a linear system with a time-varying delay and polytopic-type uncertainties. In order to construct a parameter-dependent Lyapunov functional for the system, we first devised a new method of dealing with a time-delay system without uncertainties. In this method, the derivative terms of the state, which is in the derivative of the Lyapunov functional, are retained and some free weighting matrices are used to express the relationships among the system variables, and among the terms in the Leibniz-Newton formula. As a result, the Lyapunov matrices are not involved in any product terms of the system matrices in the derivative of the Lyapunov functional. This method is then easily extended to a system with polytopic-type uncertainties. Numerical examples demonstrate the validity of the proposed criteria.**

***Index Terms*— time-varying delay, robust stability, parameter-dependent Lyapunov functional, polytopic-type uncertainties, linear matrix inequality (LMI).**

I. INTRODUCTION

TIME-DELAY systems are frequently encountered in various areas, including engineering, biology, and economics (see [1]). A time delay is often a source of instability and oscillations in a system. In the past few years, the robust stability of uncertain systems with time delays has received considerable attention; and many papers have focused on time-delay systems with polytopic-type uncertainties. Recent efforts have shown that a parameter-dependent Lyapunov function/functional can overcome the conservatism of quadratic stability conditions (see [2]–[10]). On the other hand, current efforts to achieve robust stability in time-delay systems can be divided into two categories (e.g., [11]), namely delay-independent criteria (see [12]) and delay-dependent criteria (see [7]–[10], [13]–[24]). It is well known that delay-independent criteria tend to be conservative, especially when the size of a delay is small. Recently, Park [18] presented an improved version of the standard bounding method and obtained some delay-dependent criteria for linear time-delay systems that were better than previous results. Moon *et al.* [19] extended Park's idea to a more general form for uncertain systems with time-invariant delays. Fridman and Shaked [8]–[10] combined Park's and Moon's inequalities with a descriptor model transform and obtained more efficient criteria for systems with polytopic-type uncertainties. Even though these efforts have produced great progress, some issues remain that require reconsideration. To give an example, in the derivative of the Lyapunov functional, they used the Leibniz-Newton formula and replaced the term $x(t-\tau)$ with $x(t)-\int_{t-\tau}^{t}\dot{x}(s)ds$ in some places, but retained it in other places in order to make the calculations easier. More specifically, in [19], $x(t-\tau)$ is replaced by $x(t)-\int_{t-\tau}^{t}\dot{x}(s)ds$ in the term $2x^T(t)PA_1\dot{x}(t)$, but not in the term $\tau\dot{x}^T(t)Z\dot{x}(t)$. Since both $x(t-\tau)$ and $x(t)-\int_{t-\tau}^{t}\dot{x}(s)ds$ affect the results, there must be a relationship between them; but this point was not considered.

This paper presents some simple delay-dependent stability criteria for linear systems with a time-varying delay. First, we deal with a system with a time-varying delay that has fixed system matrices. (System matrices are the matrices in the dynamic equation of the system.) In the derivative of the Lyapunov functional, the term $\dot{x}(t)$ is retained, but the relationship among the terms in the system equation is expressed by some free weighting matrices. In consequence, the Lyapunov matrices, which are the matrices in the Lyapunov functional, are not involved in any product terms with the system matrices. Moreover, the relationship between $x(t)$, $x(t-d(t))$ and $\int_{t-d(t)}^{t}\dot{x}(s)ds$ is expressed in terms of free weighting matrices. This treatment avoids difficulties in the handling of the Lyapunov functional. The results are expressed as LMIs [25], and all of the parameters can easily be obtained numerically. Then, this idea is extended to a time-varying-delay system with polytopic-type uncertainties, and a less conservative criterion is obtained. On the other hand, it is shown that the new criterion includes the delay-independent/rate-dependent, delay-dependent/rate-independent, and delay- and rate-independent criteria as special cases. Numerical examples show that the results obtained in this paper are effective and are an improvement over existing criteria.

Y. He and M. Wu are with the School of Information Science and Engineering, Central South University, Changsha 410083, China.

J.-H. She is with the School of Bionics, Tokyo University of Technology, Tokyo, 192-0982 Japan.

G.-P. Liu is with the School of Mechanical, Materials, Manufacturing Engineering and Management, University of Nottingham, Nottingham, NG7 2RD, UK, and is also with the Institute of Automation, Chinese Academy of Sciences, Beijing 100080, China.

II. STABILITY ISSUES

Consider a linear system Σ with a time-varying delay

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Bx(t-d(t)),\ t>0, \\ x(t) = \phi(t),\ t \in [-\tau, 0], \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state vector. The matrices A and B are subject to uncertainties and satisfy the real convex polytopic model

$$[A\ B] \in \Omega, \quad \Omega := \left\{ [A(\xi)\ B(\xi)] = \sum_{j=1}^{p} \xi_j [A_j\ B_j],\ \sum_{j=1}^{p} \xi_j = 1,\ \xi_j \geq 0 \right\}, \tag{2}$$

where A_j, B_j $(j = 1, \cdots, p)$ are constant matrices with appropriate dimensions and ξ_j $(j = 1, \cdots, p)$ are time-invariant uncertainties. The time delay $d(t)$ is a time-varying continuous function that satisfies

$$0 \leq d(t) \leq \tau, \tag{3}$$

and

$$\dot{d}(t) \leq \mu < 1, \tag{4}$$

where τ and μ are constants and the initial condition $\phi(t)$ denotes a continuous vector-valued initial function of $t \in [-\tau, 0]$.

A. Delay-dependent Asymptotic Stability

In order to discuss the stability of System Σ, which has polytopic-type uncertainties (2), first, we consider the case in which the matrices A and B are fixed, i.e., the system has no uncertainties. For this case, the following lemma holds.

Lemma 1: For given scalars $\tau > 0$ and $\mu < 1$, System Σ with fixed matrices A and B and a time-varying delay satisfying (3) and (4) is asymptotically stable if there exist $P = P^T > 0$, $Q = Q^T \geq 0$, $Z = Z^T > 0$ and appropriately dimensioned matrices N_i and T_i $(i = 1, 2, 3)$ such that the following LMI holds:

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau N_1 \\ \Gamma_{12}^T & \Gamma_{22} & \Gamma_{23} & \tau N_2 \\ \Gamma_{13}^T & \Gamma_{23}^T & \Gamma_{33} & \tau N_3 \\ \tau N_1^T & \tau N_2^T & \tau N_3^T & -\tau Z \end{bmatrix} < 0, \tag{5}$$

where

$$\begin{aligned} \Gamma_{11} &= Q + N_1 + N_1^T - A^T T_1^T - T_1 A, \\ \Gamma_{12} &= N_2^T - N_1 - A^T T_2^T - T_1 B, \\ \Gamma_{13} &= P + N_3^T + T_1 - A^T T_3^T, \\ \Gamma_{22} &= -(1-\mu)Q - N_2 - N_2^T - T_2 B - B^T T_2^T, \\ \Gamma_{23} &= -N_3^T + T_2 - B^T T_3^T, \\ \Gamma_{33} &= \tau Z + T_3 + T_3^T. \end{aligned}$$

Proof: Choose a Lyapunov functional candidate to be

$$\begin{aligned} V(x_t) \;\; := \;\; & x^T(t)Px(t) + \int_{t-d(t)}^{t} x^T(s)Qx(s)ds \\ & + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta, \end{aligned} \tag{6}$$

where matrices $P = P^T > 0$, $Q = Q^T \geq 0$ and $Z = Z^T \geq 0$ need to be determined. Calculating the derivative of $V(x_t)$ along the solution of System Σ yields

$$\begin{aligned} \dot{V}(x_t) &= 2x^T(t)P\dot{x}(t) \\ &\quad + x^T(t)Qx(t) - (1-\dot{d}(t))x^T(t-d(t))Qx(t-d(t)) \\ &\quad + \tau\dot{x}^T(t)Z\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}^T(s)Z\dot{x}(s)ds \\ &\leq 2x^T(t)P\dot{x}(t) \\ &\quad + x^T(t)Qx(t) - (1-\mu)x^T(t-d(t))Qx(t-d(t)) \\ &\quad + \tau\dot{x}^T(t)Z\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s)Z\dot{x}(s)ds. \end{aligned} \tag{7}$$

The Leibniz-Newton formula provides

$$x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds = 0. \tag{8}$$

So, it is clear that, for appropriately dimensioned matrices N_i $(i = 1, 2, 3)$, the following is true:

$$\begin{aligned} & 2\left[x^T(t)N_1 + x^T(t-d(t))N_2 + \dot{x}^T(t)N_3\right] * \\ & \left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds\right] = 0. \end{aligned} \tag{9}$$

Moreover, according to Eq. (1), for appropriately dimensioned matrices T_i $(i = 1, 2, 3)$, we have

$$\begin{aligned} & 2\left[x^T(t)T_1 + x^T(t-d(t))T_2 + \dot{x}^T(t)T_3\right] * \\ & [\dot{x}(t) - Ax(t) - Bx(t-d(t))] = 0. \end{aligned} \tag{10}$$

On the other hand, for a semi-positive definite matrix $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0$, the following holds.

$$\tau\eta^T(t)X\eta(t) - \int_{t-d(t)}^{t} \eta^T(t)X\eta(t)ds \geq 0, \tag{11}$$

where

$$\eta(t) = [x^T(t) \;\; x^T(t-d(t)) \;\; \dot{x}^T(t)]^T.$$

Then, adding the terms on the left of Eqs. (9)-(11) to $\dot{V}(x_t)$ allows us to express $\dot{V}(x_t)$ as

$$\dot{V}(x_t) \leq \eta^T(t)\Xi\eta(t) - \int_{t-d(t)}^{t} \zeta^T(t,s)\Psi\zeta(t,s)ds, \tag{12}$$

where

$$\zeta(t,s) = [\eta^T(t),\ \dot{x}^T(s)]^T,$$

$$\Xi = \begin{bmatrix} \Gamma_{11} + \tau X_{11} & \Gamma_{12} + \tau X_{12} & \Gamma_{13} + \tau X_{13} \\ \Gamma_{12}^T + \tau X_{12}^T & \Gamma_{22} + \tau X_{22} & \Gamma_{23} + \tau X_{23} \\ \Gamma_{13}^T + \tau X_{13}^T & \Gamma_{23}^T + \tau X_{23}^T & \Gamma_{33} + \tau X_{33} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ X_{12}^T & X_{22} & X_{23} & N_2 \\ X_{13}^T & X_{23}^T & X_{33} & N_3 \\ N_1^T & N_2^T & N_3^T & Z \end{bmatrix}. \tag{13}$$

If $\Xi < 0$ and $\Psi \geq 0$, then $\dot{V}(x_t) < -\varepsilon\|x(t)\|^2$ for a sufficiently small ε, which ensures the asymptotic stability of System Σ [26]. Specifically, if we select a $Z > 0$ then an X can be chosen to be $X = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} Z^{-1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}^T$.

This ensures that $X \geq 0$ and $\Psi \geq 0$. In this case, $\Xi < 0$ is equivalent to $\Gamma < 0$, according to the Schur complement [25]. ■

Remark 1: It is clear from the proof of Lemma 1 that the free weighting matrices T_i $(i = 1, 2, 3)$ in Eq. (10) are used to express the relationship between the term $\dot{x}(t)$ and terms $x(t)$ and $x(t - d(t))$. The zero term $2[x^T(t)T_1 + x^T(t - d(t))T_2 + \dot{x}^T(t)T_3]\,[\dot{x}(t) - Ax(t) - Bx(t - d(t))]$ is inserted into the derivative of the Lyapunov functional so that the LMI (5), which determines the stability of the system, does not include any terms containing the product of the Lyapunov matrices and the system matrices. This idea can easily be extended to a parameter-dependent Lyapunov functional for a system with polytopic-type uncertainties (2). Moreover, the Leibniz-Newton formula (9) is also employed to make the criterion delay-dependent. Since the free weighting matrices N_i $(i = 1, 2, 3)$ in Eq. (9) express the relationship among $x(t)$, $x(t - d(t))$ and $\int_{t-d(t)}^{t} \dot{x}(s)ds$, the relationship among the terms in the Leibniz-Newton formula is taken into account. In addition, the optimal weighting matrices T_i and N_i $(i = 1, 2, 3)$ can easily be determined by solving LMI (5).

As shown in the proof of Lemma 1, Ψ in (13) can be chosen to be semi-positive, and the matrices $T_2,\ N_2,\ X_{12},\ X_{22}$ and X_{23} in Ξ and Ψ in (13) provide some extra freedom in the selection of the weighting matrices, which have the potential to yield less conservative results. When they are all zero, we obtain the following corollary, which is equivalent to Lemma 1 in [8] for systems with a single delay.

Corollary 1: For given scalars $\tau > 0$ and $\mu < 1$, System Σ with fixed matrices A and B and a time-varying delay satisfying (3) and (4) is asymptotically stable if there exist $P = P^T > 0$, $Q = Q^T \geq 0$, $Z = Z^T \geq 0$, $X = \begin{bmatrix} X_{11} & X_{13} \\ X_{13}^T & X_{33} \end{bmatrix} \geq 0$, and appropriately dimensioned matrices $N_1,\ N_3,\ T_1$ and T_3 such that the following LMIs hold.

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} \end{bmatrix} < 0, \tag{14}$$

$$\Lambda = \begin{bmatrix} X_{11} & X_{13} & N_1 \\ X_{13}^T & X_{33} & N_3 \\ N_1^T & N_3^T & Z \end{bmatrix} \geq 0, \tag{15}$$

where

$$\begin{aligned}
\Pi_{11} &= Q + N_1 + N_1^T - A^T T_1^T - T_1 A + \tau X_{11},\\
\Pi_{12} &= -N_1 - T_1 B,\\
\Pi_{13} &= P + N_3^T + T_1 - A^T T_3^T + \tau X_{13},\\
\Pi_{22} &= -(1-\mu)Q,\\
\Pi_{23} &= -N_3^T - B^T T_3^T,\\
\Pi_{33} &= \tau Z + T_3 + T_3^T + \tau X_{33}.
\end{aligned}$$

B. Delay-dependent Robust Stability

It is clear that there do not exist any terms containing the product of any combination of P, Q and Z, or AB in the derivative of the Lyapunov functional in Lemma 1. Therefore, this method can easily be extended to provide an LMI-based delay-dependent robust stability condition for System Σ with polytopic-type uncertainties as follows:

Theorem 1: For given scalars $\tau > 0$ and $\mu < 1$, System Σ with polytopic-type uncertainties (2) and a time-varying delay satisfying (3) and (4) is robustly stable if there exist $P_j = P_j^T > 0$, $Q_j = Q_j^T \geq 0$ and $Z_j = Z_j^T > 0$ $(j = 1, \cdots, p)$, and appropriately dimensioned matrices N_{ij} $(i = 1, 2, 3;\ j = 1, \cdots, p)$ and T_i $(i = 1, 2, 3)$ such that the following LMIs hold for $j = 1, \cdots, p$:

$$\bar{\Gamma}^{(j)} = \begin{bmatrix} \bar{\Gamma}_{11}^{(j)} & \bar{\Gamma}_{12}^{(j)} & \bar{\Gamma}_{13}^{(j)} & \tau N_{1j} \\ (\bar{\Gamma}_{12}^{(j)})^T & \bar{\Gamma}_{22}^{(j)} & \bar{\Gamma}_{23}^{(j)} & \tau N_{2j} \\ (\bar{\Gamma}_{13}^{(j)})^T & (\bar{\Gamma}_{23}^{(j)})^T & \bar{\Gamma}_{33}^{(j)} & \tau N_{3j} \\ \tau N_{1j}^T & \tau N_{2j}^T & \tau N_{3j}^T & -\tau Z_j \end{bmatrix} < 0, \tag{16}$$

where

$$\begin{aligned}
\bar{\Gamma}_{11}^{(j)} &= Q_j + N_{1j} + N_{1j}^T - A_j^T T_1^T - T_1 A_j,\\
\bar{\Gamma}_{12}^{(j)} &= N_{2j}^T - N_{1j} - A_j^T T_2^T - T_1 B_j,\\
\bar{\Gamma}_{13}^{(j)} &= P_j + N_{3j}^T + T_1 - A_j^T T_3^T,\\
\bar{\Gamma}_{22}^{(j)} &= -(1-\mu)Q_j - N_{2j} - N_{2j}^T - T_2 B_j - B_j^T T_2^T,\\
\bar{\Gamma}_{23}^{(j)} &= -N_{3j}^T + T_2 - B_j^T T_3^T,\\
\bar{\Gamma}_{33}^{(j)} &= \tau Z_j + T_3 + T_3^T.
\end{aligned}$$

Proof: Choose a Lyapunov functional candidate to be

$$\begin{aligned}
V_u(x_t) \;:=\; & \sum_{j=1}^{p} x^T(t)\xi_j P_j x(t) \\
& + \sum_{j=1}^{p} \int_{t-d(t)}^{t} x^T(s)\xi_j Q_j x(s)ds \\
& + \sum_{j=1}^{p} \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)\xi_j Z_j \dot{x}(s)dsd\theta,
\end{aligned} \tag{17}$$

where $P_j = P_j^T > 0$, $Q_j = Q_j^T \geq 0$ and $Z_j = Z_j^T \geq 0$ $(j = 1, \cdots, p)$ need to be determined. As in Lemma 1, the derivative of $V_u(x_t)$ along the solutions of System Σ can be expressed as

$$\begin{aligned}
\dot{V}_u(x_t) \;\leq\; & \sum_{j=1}^{p} \eta^T(t)\xi_j \bar{\Xi}^{(j)} \eta(t) \\
& - \sum_{j=1}^{p} \int_{t-d(t)}^{t} \zeta^T(t,s)\xi_j \bar{\Psi}^{(j)} \zeta(t,s)ds,
\end{aligned} \tag{18}$$

where $\eta(t)$ and $\zeta(t,s)$ are defined in Lemma 1, and

$$\bar{\Xi}^{(j)} = \begin{bmatrix} \bar{\Xi}_{11}^{(j)} & \bar{\Xi}_{12}^{(j)} & \bar{\Xi}_{13}^{(j)} \\ (\bar{\Xi}_{12}^{(j)})^T & \bar{\Xi}_{22}^{(j)} & \bar{\Xi}_{23}^{(j)} \\ (\bar{\Xi}_{13}^{(j)})^T & (\bar{\Xi}_{23}^{(j)})^T & \bar{\Xi}_{33}^{(j)} \end{bmatrix}, \tag{19}$$

$$\bar{\Xi}_{ik}^{(j)} = \bar{\Gamma}_{ik}^{(j)} + \tau X_{ik}^{(j)} \;(i = 1, 2, 3; i \leq k \leq 3), \tag{20}$$

$$\bar{\Psi}^{(j)} = \begin{bmatrix} X_{11}^{(j)} & X_{12}^{(j)} & X_{13}^{(j)} & N_{1j} \\ (X_{12}^{(j)})^T & X_{22}^{(j)} & X_{23}^{(j)} & N_{2j} \\ (X_{13}^{(j)})^T & (X_{23}^{(j)})^T & X_{33}^{(j)} & N_{3j} \\ N_{1j}^T & N_{2j}^T & N_{3j}^T & Z_j \end{bmatrix}. \tag{21}$$

So, following the same statements in Lemma 1, $\bar{\Xi}^{(j)} < 0$ and $\bar{\Psi}^{(j)} \geq 0$ ensure the robust stability of System Σ. As a special case, if we choose $Z_j > 0$ and set $X^{(j)} = \begin{bmatrix} N_{1j} \\ N_{2j} \\ N_{3j} \end{bmatrix} Z_j^{-1} \begin{bmatrix} N_{1j} \\ N_{2j} \\ N_{3j} \end{bmatrix}^T$ in Eqs. (20) and (21), then the conditions $\bar{\Xi}^{(j)} < 0$ and $\bar{\Psi}^{(j)} \geq 0$ are equivalent to LMIs (16). ■

Theorem 1 gives a delay- and rate-dependent robust criterion for a delay satisfying (3) and (4). Note that a delay-dependent and rate-independent criterion for a delay satisfying (3) can be derived from Theorem 1 by choosing $Q_j = 0$ $(j = 1, \cdots, p)$ as follows.

Corollary 2: For a given scalar $\tau > 0$, System Σ with polytopic-type uncertainties (2) and a time-varying delay satisfying (3) is robustly stable if there exist $P_j = P_j^T > 0$ and $Z_j = Z_j^T > 0$ $(j = 1, \cdots, p)$, and appropriately dimensioned matrices N_{ij} $(i = 1, 2, 3;\ j = 1, \cdots, p)$ and T_i $(i = 1, 2, 3)$ such that the LMIs (22) hold for $j = 1, \cdots, p$.

$$\hat{\Gamma}^{(j)} = \begin{bmatrix} \hat{\Gamma}_{11}^{(j)} & \bar{\Gamma}_{12}^{(j)} & \bar{\Gamma}_{13}^{(j)} & \tau N_{1j} \\ (\bar{\Gamma}_{12}^{(j)})^T & \hat{\Gamma}_{22}^{(j)} & \bar{\Gamma}_{23}^{(j)} & \tau N_{2j} \\ (\bar{\Gamma}_{13}^{(j)})^T & (\bar{\Gamma}_{23}^{(j)})^T & \bar{\Gamma}_{33}^{(j)} & \tau N_{3j} \\ \tau N_{1j}^T & \tau N_{2j}^T & \tau N_{3j}^T & -\tau Z_j \end{bmatrix} < 0, \quad (22)$$

where

$$\begin{aligned} \hat{\Gamma}_{11}^{(j)} &= N_{1j} + N_{1j}^T - A_j^T T_1^T - T_1 A_j, \\ \hat{\Gamma}_{22}^{(j)} &= -N_{2j} - N_{2j}^T - T_2 B_j - B_j^T T_2^T, \end{aligned}$$

and $\bar{\Gamma}_{12}^{(j)}$, $\bar{\Gamma}_{13}^{(j)}$, $\bar{\Gamma}_{23}^{(j)}$ and $\bar{\Gamma}_{33}^{(j)}$ are defined in (16).

C. Delay-independent Robust Stability

As shown in the proof of Theorem 1, $\bar{\Psi}^{(j)}$ only has to be semi-positive, rather than positive. So, if we set the matrices Z_j, $X^{(j)}$ $(j = 1, \cdots, p)$ and N_{ij} $(i = 1, 2, 3;\ j = 1, \cdots, p)$ to zero, then we can obtain a delay-independent and rate-dependent criterion. In this case, Theorem 1 becomes the following corollary.

Corollary 3: For a given scalar $\mu < 1$, System Σ with polytopic-type uncertainties (2) and a time-varying delay satisfying (4) is robustly stable if there exist $P_j = P_j^T > 0$ and $Q_j = Q_j^T \geq 0$ $(j = 1, \cdots, p)$, and appropriately dimensioned matrices T_i $(i = 1, 2, 3)$ such that the following LMIs hold for $j = 1, \cdots, p$:

$$\Phi^{(j)} = \begin{bmatrix} \Phi_{11}^{(j)} & \Phi_{12}^{(j)} & \Phi_{13}^{(j)} \\ (\Phi_{12}^{(j)})^T & \Phi_{22}^{(j)} & \Phi_{23}^{(j)} \\ (\Phi_{13}^{(j)})^T & (\Phi_{23}^{(j)})^T & \Phi_{33}^{(j)} \end{bmatrix} < 0, \quad (23)$$

where

$$\begin{aligned} \Phi_{11}^{(j)} &= Q_j - A_j^T T_1^T - T_1 A_j, \\ \Phi_{12}^{(j)} &= -A_j^T T_2^T - T_1 B_j, \\ \Phi_{13}^{(j)} &= P_j + T_1 - A_j^T T_3^T, \\ \Phi_{22}^{(j)} &= -(1-\mu) Q_j - T_2 B_j - B_j^T T_2^T, \\ \Phi_{23}^{(j)} &= T_2 - B_j^T T_3^T, \\ \Phi_{33}^{(j)} &= T_3 + T_3^T. \end{aligned}$$

If a system is seen to be stable based on the delay-independent criterion in Corollary 3, then the system is robustly stable for a time-varying delay, $d(t)$, of any size that satisfies (4) in the system, according to Theorem 1.

In addition, a delay- and rate-independent criterion also can be derived from Corollary 3 by choosing $Q_j = 0$ $(j = 1, \cdots, p)$.

Corollary 4: System Σ with polytopic-type uncertainties (2) is robustly stable if there exist $P_j = P_j^T > 0$ $(j = 1, \cdots, p)$ and appropriately dimensioned matrices T_i $(i = 1, 2, 3)$ such that the following LMIs hold for $j = 1, \cdots, p$.

$$\bar{\Phi}^{(j)} = \begin{bmatrix} -A_j^T T_1^T - T_1 A_j & \Phi_{12}^{(j)} & \Phi_{13}^{(j)} \\ (\Phi_{12}^{(j)})^T & -T_2 B_j - B_j^T T_2^T & \Phi_{23}^{(j)} \\ (\Phi_{13}^{(j)})^T & (\Phi_{23}^{(j)})^T & \Phi_{33}^{(j)} \end{bmatrix} < 0, \quad (24)$$

where $\Phi_{12}^{(j)}, \Phi_{13}^{(j)}, \Phi_{23}^{(j)}$ and $\Phi_{33}^{(j)} (j = 1, \cdots, p)$ are defined in (23).

III. Numerical Examples

This section provides two examples that demonstrate how effective the criteria presented in this paper are and that they are an improvement over existing methods.

Example 1: Consider System Σ with polytopic-type uncertainties [7]:

$$A_1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, \ A_2 = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ B_3 = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}.$$

When $\mu = 0$, the delay is time-invariant. The upper bound on the time-delay was found to be 0.4149 in [7] and 4.2423 in [8]–[10]. However, based on Theorem 1 in this paper, the system is robustly stable for $\tau = 4.2501$. The maximum upper bound on the allowable size of the time delay given by Theorem 1 is more than 9 times larger than the one in [7], and is better than the one in [8]–[10]. Xia's method [7] cannot handle the case $\mu \neq 0$. Table I shows a comparison of the upper bounds for $\mu \neq 0$ obtained by Fridman and Shaked's method [8]–[10] and by our methods. (Theorem 1 and Corollary 2 were used to calculate the cases $\mu = 0 \sim 0.9$ and any μ (rate-independent), respectively.) It is clear that, since the relationships between the term $\dot{x}(t)$ and terms $x(t)$ and $x(t - d(t))$, and between $x(t)$, $x(t - d(t))$ and $\int_{t-d(t)}^{t} \dot{x}(s)ds$ are taken into account, the upper bounds obtained in this paper are larger than those in [8]–[10].

Example 2: Consider the following System Σ with a time-varying delay

$$A = \begin{bmatrix} 0 & -0.12 + 12\rho \\ 1 & -0.465 - \rho \end{bmatrix}, \ B = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}, \quad (25)$$

and $|\rho| \leq 0.035$ [10].

TABLE I

CALCULATION RESULTS FOR EXAMPLE 1.

μ	0	0.1	0.5	0.9	any μ
Xia [7]	0.4149	-	-	-	-
Fridman and Shaked [8]–[10]	4.2423	3.3555	1.8088	0.9670	0.7963
Our method	4.2501	3.3637	1.8261	1.0589	0.9090

If we let $\rho_m = 0.035$ and set

$$A_1 = \begin{bmatrix} 0 & -0.12+12\rho_m \\ 1 & -0.465-\rho_m \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & -0.12-12\rho_m \\ 1 & -0.465+\rho_m \end{bmatrix}, \\ B_1 = B_2 = B = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}, \tag{26}$$

then the system is described by Eq. (2). When $\mu = 0$, the upper bound on the time-delay obtained in [8] and [9] is 0.782. Theorem 1 in [10] yields an upper bound of 0.863, according to that paper, although calculations using MATLAB 6.5 and LMI Control Toolbox 1.0.8 yield a value of 0.782. From Theorem 1 in this paper, we found that the system is robustly stable for $\tau = 0.863$, which is better than the values in [8], [9] and the same as the one in [10]. In addition, Table II shows a comparison of the upper bounds for $\mu \neq 0$ obtained by Fridman and Shaked's method [8]–[10] and by our methods. (Theorem 1 and Corollary 2 were used to calculate the cases $\mu = 0 \sim 0.9$ and any μ (rate-independent), respectively.) It is clear that the upper bounds obtained in this paper are larger than, or at least equal to, those given in [8]–[10].

TABLE II

CALCULATION RESULTS FOR EXAMPLE 2.

μ	0	0.1	0.5	0.9	any μ
Fridman and Shaked [8]–[10]	0.782	0.736	0.465	0.454	0.454
Our method	0.863	0.786	0.465	0.454	0.454

IV. CONCLUSION

This paper presents some new stability criteria for time-delay systems with polytopic-type uncertainties. New techniques were developed to make the criteria less conservative. First, a delay-dependent stability criterion for a system without uncertainties was derived by considering the relationships among the terms in the system variables. The relationships among the terms in the Leibniz-Newton formula were also taken into account using some free weighting matrices, which were selected by LMIs. These techniques eliminated the terms containing the product of the Lyapunov matrices and the system matrices in the derivative of the Lyapunov functional. Then, this idea was extended to a system with polytopic-type uncertainties, and new stability criteria were obtained. Two numerical examples demonstrate the validity of the these methods. The results show that the methods described in this paper are very effective and are an improvement over existing methods.

REFERENCES

[1] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach (Lecture Notes in Control and Information Sciences, 269)*. New York: Springer-Verlag, 2001.

[2] J. C. Geromel, M. C. de Oliveira, and L. Hsu, "LMI characterization of structural and robust stability," *Linear Algebra and its Applications*, vol. 285, pp. 69–80, 1998.

[3] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Systems and Control Letters*, vol. 37, no. 4, pp. 261–265, 1999.

[4] M. C. de Oliveira, J. Bernussou, and L. Hsu, "LMI characterization of structural and robust stability: the discrete-time case," *Linear Algebra and its Applications*, vol. 296, pp. 27–38, 1999.

[5] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, "A new robust D-stability condition for real convex polytopic uncertainty," *Systems and Control Letters*, vol. 40, no. 1, pp. 21–30, 2000.

[6] U. Shaked, "Improved LMI representations for analysis and design of continuous-time systems with polytopic type uncertainty," *IEEE Trans. Automat. Contr.*, vol. 46, no. 4, pp. 652–656, 2001.

[7] Y. Q. Xia and Y. M. Jia, "Robust stability functionals of state delayed systems with polytopic type uncertainties via parameter-dependent lyapunov functions," *Int. J. Control*, vol. 75, no. 16, pp. 1427–1434, 2002.

[8] E. Fridman and U. Shaked, "An improved stabilization method for linear time-delay systems," *IEEE Trans. Automat. Contr.*, vol. 47, no. 11, pp. 1931–1937, 2002.

[9] ——, "Delay-dependent stability and H_∞ control: constant and time-varying delays," *Int. J. Control*, vol. 76, no. 1, pp. 48–60, 2003.

[10] ——, "Parameter dependent stability and stabilization of uncertain time-delay systems," *IEEE Trans. Automat. Contr.*, vol. 48, no. 5, pp. 861–866, 2003.

[11] T. Mori, "Criteria for asymptotic stability of linear time-delay systems," *IEEE Trans. Automat. Contr.*, vol. 30, no. 2, pp. 158–161, 1985.

[12] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless H_∞ controllers for state delayed systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 159–162, 1994.

[13] T. J. Su and C. G. Huang, "Robust stability of delay dependence for linear uncertain systems," *IEEE Trans. Automat. Contr.*, vol. 37, no. 10, pp. 1656–1659, 1992.

[14] X. Li and C. E. de Souza, "Delay-dependent robust stability and stabilization of uncertain linear dealy systems: A linear matrix inequality approach," *IEEE Trans. Automat. Contr.*, vol. 42, no. 8, pp. 1144–1148, 1997.

[15] Y. Gu, S. Wang, Z. C. Q. Li, and J. Qian, "On delay-dependent stability and decay estimate for uncertain systems with time-varying delay," *Automatica*, vol. 34, no. 8, pp. 1035–1039, 1998.

[16] Y. Y. Cao, Y. X. Sun, and C. W. Cheng, "Delay-dependent robust stabilization of uncertain systems with multiple state delays," *IEEE Trans. Automat. Contr.*, vol. 43, no. 11, pp. 1608–1612, 1998.

[17] C. E. de Souza and X. Li, "Delay-dependent robust H_∞ control of uncertain linear state-delayed systems," *Automatica*, vol. 35, no. 7, pp. 1313–1321, 1999.

[18] P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays," *IEEE Trans. Automat. Contr.*, vol. 44, no. 4, pp. 876–877, 1999.

[19] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems," *Int. J. Control*, vol. 74, no. 14, pp. 1447–1455, 2001.

[20] Q. L. Han and K. Q. Gu, "On robust stability of time-delay systems with norm-bounded uncertainty," *IEEE Trans. Automat. Contr.*, vol. 46, no. 9, pp. 1426–1431, 2001.

[21] J. H. Kim, "Delay and its time-derivative dependent roubst stability of time-delayed linear systems with uncertainty," *IEEE Trans. Automat. Contr.*, vol. 46, no. 5, pp. 789–792, 2001.

[22] Q. L. Han, "New results for delay-dependent stability of linear systems with time-varying delay," *Int. J. Sys. Sci.*, vol. 33, pp. 213–228, 2002.

[23] ——, "Robust stability of uncertain delay-differential systems of neutral type," *Automatica*, vol. 38, no. 4, pp. 719–723, 2002.

[24] D. Yue and S. Won, "An improvement on 'delay and its time-derivative dependent roubst stability of time-delayed linear systems with uncertainty'," *IEEE Trans. Automat. Contr.*, vol. 47, no. 2, pp. 407–408, 2002.

[25] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequality in system and control theory*. Philadelphia: SIAM, 1994.

[26] J. Hale, *Theory of functional differential equations*. New York: Springer-Verlag, 1977.