IEE Proceedings - Control Theory and Applications -, Vol. 153, No. 4, pp. 447-452, July, 2006.

Delay-Dependent Stability Criteria for Linear Systems with Multiple Time Delays *

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Abstract

This paper deals with the problem of the delay-dependent stability of linear systems with multiple time delays. A new method is first presented for a system with two time delays, in which free weighting matrices are used to express the relationships among the terms of the Leibniz-Newton formula. Next, this method is used to show the equivalence between a system with two identical time delays and a system with a single time delay. Then, a numerical example verifies that the criterion given in this paper is effective and is a significant improvement over existing ones. Finally, the basic idea is extended to a system with multiple time delays.

Key words: delay-dependent criteria, stability, linear matrix inequality (LMI), multiple delays, free weighting matrix.

1 Introduction

The problem of deriving delay-dependent stability criteria for linear delay systems has attracted the attention of many researchers in the last decade (e.g. [1–22]). They have mainly employed a Lyapunov functional to derive delay-dependent criteria; and the

*This work was supported in part by the National Science Foundation of China under Grant Nos. 60425310 and 60574014; the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of the Ministry of Education, P. R. China; and the Doctor Subject Foundation of China under Grant No. 20050533015.

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resulting methods are generally classified into two categories: the discretized-Lyapunov-functional method and methods based on the Leibniz-Newton formula. It is difficult to extend the former type (for instance, [6, 7, 10]) to solve the synthesis problem for a control system, and the latter type generally requires a system transformation. Four basic fixed model transformations have been presented (e.g., see [5]); but they all entail a certain degree of conservativeness, which leaves room for further investigation. Recently, He *et al.* [14, 15] and Wu *et al.* [21, 22] devised a new method that employed free weighting matrices to express the relationships between the terms of the Leibniz-Newton formula. This overcomes the conservativeness of methods involving a fixed model transformation.

On the other hand, it is well known that, if a linear system with a single time delay, h, is not stable for a delay of any length but is stable for $h = 0$, then there must exist a positive number $\bar{h}$ for which the system is stable for $0 \le h \le \bar{h}$. Many researchers have simply extended this idea to a system with multiple time delays. However, this simple extension may lead to conservatism. For example, [4, 5] dealt with a linear system containing two time delays:

$$\dot{x}(t) = A_0x(t) + A_1x(t - h_1) + A_2x(t - h_2). \tag{1}$$

The upper bounds $\bar{h}_1$ and $\bar{h}_2$ on h_1 and h_2, respectively, are selected such that (1) is stable for $0 \le h_1 \le \bar{h}_1$ and $0 \le h_2 \le \bar{h}_2$. However, the ranges of h_1 and h_2 that ensure the stability of the system (1) are conservative because they are guaranteed from *zero* to the upper bound, even though it may not be necessary for them to start from *zero*. One reason for this is that the relationship between h_1 and h_2 was not taken into account in the procedure for finding the upper bounds. Specifically, the criterion for a linear system with a single delay

$$\dot{x}(t) = A_0x(t) + (A_1 + A_2)x(t - h), \tag{2}$$

should be equivalent to that for a system with two delays (1) in which $h_1 = h_2$; but this equivalence cannot be demonstrated by the methods in [4, 5].

In this paper, new delay-dependent criteria are proposed based on a new method. Concretely, a criterion is first established for a linear system with two time delays. Not only the relationships between $x(t - h_1)$ and $x(t) - \int_{t-h_1}^{t} \dot{x}(s)ds$, and $x(t - h_2)$ and $x(t) - \int_{t-h_2}^{t} \dot{x}(s)ds$, but also that between $x(t - h_2)$ and $x(t - h_1) - \int_{t-h_2}^{t-h_1} \dot{x}(s)ds$ are taken into account. Note that the last relationship is precisely that between h_1 and h_2. All these relationships are expressed in terms of weighting matrices and are determined from the solutions of linear matrix inequalities (LMIs). Furthermore, the equivalence between systems (1) and (2) for $h_1 = h_2$ is demonstrated. A numerical example is used to show that the criteria given in this paper are effective and are a significant improvement over existing ones. Finally, the basic idea is extended to a system with multiple time delays.

2 Delay-Dependent Stability

Consider the following linear system, Σ, with multiple time delays:

$$\Sigma : \begin{cases} \dot{x}(t) = \sum_{i=0}^{m} A_i x(t - h_i), \ t > 0, \\ x(t) = \phi(t), \ t \in [-h, 0], \end{cases} \tag{3}$$

where $x(t) \in R^n$ is the state vector; $h_0 = 0$ and $h_i \geq 0$ $(i = 1, 2, \cdots, m)$ are constant time delays; $h := \max\{h_1, h_2, \cdots, h_m\}$; and $A_i \in R^{n \times n}$ $(i = 0, 1, \cdots, m)$ are constant matrices. The initial condition, $\phi(t)$, denotes a continuous vector-valued initial function of $t \in [-h, 0]$.

2.1 Two Time Delays

First, let's consider the case $m = 2$. The following delay-dependent stability criterion, which takes the relationship between h_1 and h_2 into account, is established for Σ.

Theorem 1 *For given scalars $h_i \geq 0$ $(i = 1, 2)$, the system Σ is asymptotically stable if there exist symmetric positive definite matrices $P = P^T > 0$ and $Q_i = Q_i^T > 0$ $(i = 1, 2)$, symmetric semi-positive definite matrices $W_i = W_i^T \geq 0$, $X_{ii} = X_{ii}^T \geq 0$, $Y_{ii} = Y_{ii}^T \geq 0$ and $Z_{ii} = Z_{ii}^T \geq 0$ $(i = 1, 2, 3)$, and any matrices N_i, S_i, T_i $(i = 1, 2, 3)$ and X_{ij}, Y_{ij}, Z_{ij} $(1 \leq i < j \leq 3)$ such that the following LMIs hold:*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{12}^T & \Phi_{22} & \Phi_{23} \\ \Phi_{13}^T & \Phi_{23}^T & \Phi_{33} \end{bmatrix} < 0, \tag{4}$$

$$\Psi_1 = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ X_{12}^T & X_{22} & X_{23} & N_2 \\ X_{13}^T & X_{23}^T & X_{33} & N_3 \\ N_1^T & N_2^T & N_3^T & W_1 \end{bmatrix} \geq 0, \tag{5}$$

$$\Psi_2 = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & S_1 \\ Y_{12}^T & Y_{22} & Y_{23} & S_2 \\ Y_{13}^T & Y_{23}^T & Y_{33} & S_3 \\ S_1^T & S_2^T & S_3^T & W_2 \end{bmatrix} \geq 0, \tag{6}$$

$$\Psi_3 = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & kT_1 \\ Z_{12}^T & Z_{22} & Z_{23} & kT_2 \\ Z_{13}^T & Z_{23}^T & Z_{33} & kT_3 \\ kT_1^T & kT_2^T & kT_3^T & W_3 \end{bmatrix} \geq 0, \tag{7}$$

where

$$k = \begin{cases} 1; & if\ h_1 \geq h_2, \\ -1; & if\ h_1 < h_2, \end{cases}$$

and

$\Phi_{11} = PA_0 + A_0^T P + Q_1 + Q_2 + N_1 + N_1^T + S_1 + S_1^T + A_0^T HA_0 + h_1 X_{11} + h_2 Y_{11} + |h_1 - h_2| Z_{11}$,

$\Phi_{12} = PA_1 - N_1 + N_2^T + S_2^T - T_1 + A_0^T HA_1 + h_1 X_{12} + h_2 Y_{12} + |h_1 - h_2| Z_{12}$,

$\Phi_{13} = PA_2 + N_3^T + S_3^T - S_1 + T_1 + A_0^T HA_2 + h_1 X_{13} + h_2 Y_{13} + |h_1 - h_2| Z_{13}$,

$\Phi_{22} = -Q_1 - N_2 - N_2^T - T_2 - T_2^T + A_1^T HA_1 + h_1 X_{22} + h_2 Y_{22} + |h_1 - h_2| Z_{22}$,

$\Phi_{23} = -N_3^T - S_2 + T_2 - T_3^T + A_1^T HA_2 + h_1 X_{23} + h_2 Y_{23} + |h_1 - h_2| Z_{23}$,

$\Phi_{33} = -Q_2 - S_3 - S_3^T + T_3 + T_3^T + A_2^T HA_2 + h_1 X_{33} + h_2 Y_{33} + |h_1 - h_2| Z_{33}$,

$H = h_1 W_1 + h_2 W_2 + |h_1 - h_2| W_3$.

Proof. First, let's consider the case $h_1 \geq h_2$. Choose the following candidate Lyapunov-Krasovskii functional:

$$\begin{aligned} V_2(x_t) \;:=\; & x^T(t)Px(t) + \int_{t-h_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-h_2}^{t} x^T(s)Q_2x(s)ds \\ & + \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)W_1\dot{x}(s)dsd\theta + \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)W_2\dot{x}(s)dsd\theta \\ & + \int_{-h_1}^{-h_2}\int_{t+\theta}^{t} \dot{x}^T(s)W_3\dot{x}(s)dsd\theta, \end{aligned} \tag{8}$$

where $P = P^T > 0$, $Q_i = Q_i^T > 0$ $(i = 1, 2)$, and $W_i = W_i^T \geq 0$ $(i = 1, 2, 3)$ are to be determined. Calculating the derivative of $V_2(x_t)$ along the solutions of Σ yields

$$\begin{aligned}\dot{V}_2(x_t) \quad = \quad & 2x^T(t)P\left[A_0x(t) + A_1x(t - h_1) + A_2x(t - h_2)\right] \\ & +x^T(t)Q_1x(t) - x^T(t - h_1)Q_1x(t - h_1) \\ & +x^T(t)Q_2x(t) - x^T(t - h_2)Q_2x(t - h_2) \\ & +h_1\dot{x}^T(t)W_1\dot{x}(t) - \int_{t-h_1}^{t} \dot{x}^T(s)W_1\dot{x}(s)ds \\ & +h_2\dot{x}^T(t)W_2\dot{x}(t) - \int_{t-h_2}^{t} \dot{x}^T(s)W_2\dot{x}(s)ds \\ & +(h_1 - h_2)\dot{x}^T(t)W_3\dot{x}(t) - \int_{t-h_1}^{t-h_2} \dot{x}^T(s)W_3\dot{x}(s)ds.\end{aligned} \tag{9}$$

Using the Leibniz-Newton formula, it is clear that, for any matrices N_i, S_i and T_i $(i = 1, 2, 3)$, the following are true.

$$\begin{aligned} & 2\left[x^T(t)N_1 + x^T(t - h_1)N_2 + x^T(t - h_2)N_3\right] \\ & \qquad \times \left[x(t) - x(t - h_1) - \int_{t-h_1}^{t} \dot{x}(s)ds\right] = 0,\end{aligned} \tag{10}$$

$$\begin{aligned} & 2\left[x^T(t)S_1 + x^T(t - h_1)S_2 + x^T(t - h_2)S_3\right] \\ & \qquad \times \left[x(t) - x(t - h_2) - \int_{t-h_2}^{t} \dot{x}(s)ds\right] = 0,\end{aligned} \tag{11}$$

$$\begin{aligned} & 2\left[x^T(t)T_1 + x^T(t - h_1)T_2 + x^T(t - h_2)T_3\right] \\ & \qquad \times \left[x(t - h_2) - x(t - h_1) - \int_{t-h_1}^{t-h_2} \dot{x}(s)ds\right] = 0.\end{aligned} \tag{12}$$

On the other hand, for any appropriately dimensioned matrices $X_{ii} = X_{ii}^T \geq 0$, $Y_{ii} = Y_{ii}^T \geq 0$, $Z_{ii} = Z_{ii}^T \geq 0$ $(i = 1, 2, 3)$, and X_{ij}, Y_{ij}, Z_{ij} $(1 \leq i < j \leq 3)$, the following equation holds:

$$\begin{bmatrix} x(t) \\ x(t - h_1) \\ x(t - h_2) \end{bmatrix}^T \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12}^T & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{13}^T & \Lambda_{23}^T & \Lambda_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h_1) \\ x(t - h_2) \end{bmatrix} = 0, \tag{13}$$

where

$$\Lambda_{ij} = h_1(X_{ij} - X_{ij}) + h_2(Y_{ij} - Y_{ij}) + (h_1 - h_2)(Z_{ij} - Z_{ij}),\ 1 \leq i \leq j \leq 3.$$

If we add the terms on the left in (10)-(13) to $\dot{V}_2(x_t)$, then based on the fact that, for any $r \geq 0$ and any function $f(t)$

$$\int_{t-r}^{t} f(t)ds = rf(t),$$

we can write $\dot{V}_2(x_t)$ as

$$\begin{aligned} \dot{V}_2(x_t) &= \xi_1^T(t)\Phi\xi_1(t) - \int_{t-h_1}^{t} \xi_2^T(t,s)\Psi_1\xi_2(t,s)ds \\ &\quad - \int_{t-h_2}^{t} \xi_2^T(t,s)\Psi_2\xi_2(t,s)ds - \int_{t-h_1}^{t-h_2} \xi_2^T(t,s)\Psi_3\xi_2(t,s)ds, \end{aligned} \tag{14}$$

where

$$\xi_1(t) = \left[x^T(t) \;\; x^T(t-h_1) \;\; x^T(t-h_2)\right]^T, \quad \xi_2(t,s) = \left[\xi_1^T(t) \; \dot{x}^T(s)\right]^T,$$

and Φ and Ψ_i $(i = 1,2,3)$ $(k = 1$ in $\Psi_3)$ are defined in (4)-(7). If $\Phi < 0$ and $\Psi_i \geq 0\,(i = 1,2,3)$, then $\dot{V}_2(x_t) < 0$ for any $\xi_1(t) \neq 0$. So, Σ is asymptotically stable if LMIs (4)-(7) hold.

On the other hand, when $h_1 < h_2$, one candidate Lyapunov-Krasovskii functional is

$$\begin{aligned} V_2(x_t) &:= x^T(t)Px(t) + \int_{t-h_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-h_2}^{t} x^T(s)Q_2x(s)ds \\ &\quad + \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)W_1\dot{x}(s)dsd\theta + \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)W_2\dot{x}(s)dsd\theta \\ &\quad + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)W_3\dot{x}(s)dsd\theta, \end{aligned} \tag{15}$$

and Eq. (12) can be rewritten as

$$\begin{aligned} &2\left[x^T(t)T_1 + x^T(t-h_1)T_2 + x^T(t-h_2)T_3\right] \\ &\qquad \times \left[x(t-h_2) - x(t-h_1) + \int_{t-h_2}^{t-h_1} \dot{x}(s)ds\right] = 0. \end{aligned} \tag{16}$$

Then, following the procedure for the case $h_1 \geq h_2$ yields a similar result; but note that, in this case, $k = -1$ in (7). ∎

Remark 1: The main modification of the candidate Lyapunov-Krasovskii functional is that we added the last term, which contains an integral of the state with respect to h_1 and h_2. This term plays an important role. Without it, the stability is guaranteed from 0 to the upper bounds on h_1 and h_2; but with it, the stability range is from some lower (possibly non-zero) bound to the upper bound for each h_i $(i = 1,2)$. This enlarges the stability range and, as a result, reduces the conservatism. This paper employs Eqs (12) and (16) for the first time in the calculation of the derivative of the Lyapunov-Krasovskii functional. This improvement reduces the conservatism of previous treatments arising from the replacement of the term $x(t-h)$ with $x(t) - \int_{t-h}^{t} \dot{x}(s)ds$ in some places but not in others.

Remark 2: In Theorem 1, the relationships between the terms $x(t-h_1)$ and $x(t) - \int_{t-h_1}^{t} \dot{x}(s)ds$, $x(t-h_2)$ and $x(t) - \int_{t-h_2}^{t} \dot{x}(s)ds$, and $x(t-h_2)$ and $x(t-h_1) - \int_{t-h_2}^{t-h_1} \dot{x}(s)ds$ have been taken into account through the free weighting matrices N_i, S_i and T_i $(i = 1, 2, 3)$, respectively. They are determined by solving the LMIs in Theorem 1.

The criterion for the case $h_1 = h_2$ should be equivalent to a criterion for a single delay. However, this cannot be demonstrated by previous methods. In contrast, since the relationship between h_1 and h_2 is taken into account in the above theorem, the equivalence between Theorem 1 for two identical delays and a criterion for a single delay can easily be shown, as explained below.

Let's begin with a criterion for a single delay derived directly from Theorem 1.

Corollary 1 *For $m = 1$ and a given scalar $h_1 \geq 0$, the linear system Σ is asymptotically stable if there exist symmetric positive definite matrices $\bar{P} = \bar{P}^T > 0$ and $\bar{Q} = \bar{Q}^T > 0$, symmetric semi-positive definite matrices $\bar{W} = \bar{W}^T \geq 0$ and $\bar{X}_{ii} = \bar{X}_{ii}^T \geq 0$ $(i = 1, 2,$ and any matrices $\bar{X}_{12}$ and $\bar{N}_i$ $(i = 1, 2)$ such that the following LMIs hold.*

$$\begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} \\ \bar{\Phi}_{12}^T & \bar{\Phi}_{22} \end{bmatrix} < 0, \tag{17}$$

$$\begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} & \bar{N}_1 \\ \bar{X}_{12}^T & \bar{X}_{22} & \bar{N}_2 \\ \bar{N}_1^T & \bar{N}_2^T & \bar{W} \end{bmatrix} \geq 0, \tag{18}$$

where

$$\begin{aligned}
\bar{\Phi}_{11} &= \bar{P}A_0 + A_0^T\bar{P} + \bar{Q} + \bar{N}_1 + \bar{N}_1^T + A_0^T\bar{H}A_0 + h_1\bar{X}_{11}, \\
\bar{\Phi}_{12} &= \bar{P}A_1 - \bar{N}_1 + \bar{N}_2^T + A_0^T\bar{H}A_1 + h_1\bar{X}_{12}, \\
\bar{\Phi}_{22} &= -\bar{Q} - \bar{N}_2 - \bar{N}_2^T + A_1^T\bar{H}A_1 + h_1\bar{X}_{22}, \\
\bar{H} &= h_1\bar{W}.
\end{aligned}$$

We now show that Corollary 1 is equivalent to Theorem 1 for $h_1 = h_2$ when A_1 is replaced with $A_1 + A_2$ in $\bar{\Phi}_{12}$ and $\bar{\Phi}_{22}$.

If the third row and column of (4) are inserted into the second row and column, respectively, then (4) is equivalent to the following inequality:

$$\Pi = \begin{bmatrix} \Phi_{11} & \Pi_{12} & \Phi_{13} \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} \\ \Phi_{13}^T & \Pi_{23}^T & \Phi_{33} \end{bmatrix} < 0, \tag{19}$$

where

$$\Pi_{12} = PA_1 + PA_2 + N_2^T + N_3^T - N_1 + S_2^T + S_3^T - S_1 + A_0^T H(A_1 + A_2) + h_1(X_{12} + X_{13}) + h_2(Y_{12} + Y_{13}) + |h_1 - h_2|(Z_{12} + Z_{13}),$$

$$\Pi_{22} = -(Q_1+Q_2) - N_3 - N_3^T - S_3 - S_3^T - N_2 - N_2^T - S_2 - S_2^T + (A_1+A_2)^T H(A_1+A_2) + h_1(X_{22}+X_{23}+X_{23}^T+X_{33}) + h_2(Y_{22}+Y_{23}+Y_{23}^T+Y_{33}) + |h_1-h_2|(Z_{22}+Z_{23}+Z_{23}^T+Z_{33}),$$

$$\Pi_{23} = -Q_2 - S_3 - S_3^T + T_3 - N_3^T - S_2 + T_2 + (A_1 + A_2)^T H A_2 + h_1(X_{23} + X_{33}) + h_2(Y_{23} + Y_{33}) + |h_1 - h_2|(Z_{23} + Z_{33}),$$

and Φ_{11}, Φ_{13}, Φ_{33}, and H are defined in (4).

First, if LMIs (17) and (18) in Corollary 1 hold (where A_1 is replaced with A_1+A_2), the solutions can be expressed as appropriate forms of the feasible solutions of LMIs (19), (5), (6), and (7). In fact, for the feasible solutions of LMIs (17) and (18) in Corollary 1, we can set $P = \bar{P}$, $S_i = 0$ $(i = 1, 2, 3)$, $N_1 = \bar{N}_1$, $N_2 = \bar{N}_2$, $N_3 = 0$, $0 < Q_2 < \bar{Q}$, $Q_1 = \bar{Q} - Q_2$, $T_1 = -\bar{P}A_2 - A_0^T \bar{H} A_2$, $T_2 = Q_2 - (A_1 + A_2)^T \bar{H} A_2$, $T_3 = 0$, $W_1 = \bar{W}$, $W_2 = 0$, $X_{11} = \bar{X}_{11}$, $X_{12} = \bar{X}_{12}$, $X_{13} = 0$, $X_{22} = \bar{X}_{22}$, $X_{23} = 0$, $X_{33} = 0$ and $Y_{ij} = 0$ $(1 \le i \le j \le 3)$. Then, Z_{ij} $(1 \le i \le j \le 3)$ and W_3 are the feasible solutions of the following LMI:

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & T_1 \\ Z_{12}^T & Z_{22} & Z_{23} & T_2 \\ Z_{13}^T & Z_{23}^T & Z_{33} & 0 \\ T_1^T & T_2^T & 0 & W_3 \end{bmatrix} \ge 0. \qquad (20)$$

The above matrices must be the feasible solutions of LMIs (19), (5), (6), and (7). Consequently, Theorem 1 for $h_1 = h_2$ contains Corollary 1.

On the other hand, for the feasible solutions of LMIs (19), (5), (6), and (7), setting $\bar{P} = P$, $\bar{Q} = Q_1+Q_2$, $W = \bar{W}_1+\bar{W}_2$, $\bar{N}_1 = N_1+S_1$, $\bar{N}_2 = N_2+N_3+S_2+S_3$, $\bar{X}_{11} = X_{11}+Y_{11}$, $\bar{X}_{12} = X_{12}+Y_{12}+X_{13}+Y_{13}$, and $\bar{X}_{22} = X_{22}+Y_{22}+X_{23}+Y_{23}+X_{23}^T+Y_{23}^T+X_{33}+Y_{33}$ yields the feasible solutions of LMIs (17) and (18) in Corollary 1. That is, Corollary 1 contains Theorem 1 for $h_1 = h_2$. Thus, Corollary 1 and Theorem 1 are equivalent for the case $h_1 = h_2$.

2.2 Numerical Example

Consider the stability of the system Σ with $m = 2$ and

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \; A_1 = \begin{bmatrix} -1 & 0.6 \\ -0.4 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & -0.6 \\ -0.6 & 0 \end{bmatrix}. \qquad (21)$$

If $h_1 = h_2$, the system is equivalent to Σ with $m = 1$ and

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \; A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{22}$$

[4, 5] and Corollary 1 show that the system (22) is asymptotically stable only for $0 \leq h_1 \leq 4.47$. However, [4, 5] also showed that the system (21) was asymptotically stable for $0 \leq h_1 = h_2 \leq 1.64$. This result is conservative for multiple delays. The main reason for the conservatism is that they did not take the relationship between h_1 and h_2 into account. In contrast, Theorem 1 in this paper shows that (21) is asymptotically stable for $0 \leq h_1 = h_2 \leq 4.47$. Clearly, the upper bound on $h_1 = h_2$ is much larger than that in [4, 5]; and it is the same as that obtained for a single delay. Regarding the calculated range of h_2 that ensures that (21) is asymptotically stable for a given h_1, a detailed comparison of the results of our method and the method in [4, 5] is shown in Table 1; and the results are also illustrated in Figure 2.2. It is clear that our method significantly enlarges the stability domains of h_1 and h_2.

The stable range for a single delay is generally from 0 to an upper bound; so, we usually just need to find the upper bound. Since Fridman and Shaked simply extended the method for a single delay to two delays in [4, 5], they were only able to provide a stable upper bound, but not an appropriate (possibly non-zero) lower bound, for h_2. In the numerical example, their method yielded $h_1 < 2.25$; and it was impossible to find the stable range of h_2 for $h_1 \geq 2.25$. In contrast, our method employs a cross term for h_1 and h_2 (the last term of (8) and (15)) to construct a new type of Lyapunov-Krasovskii functional. Unlike existing methods, this is not a simple extension of the treatment for a single delay; and the relationship between the two delays is taken into account. Consequently, our method yields a stable range for h_2 rather than a simple upper bound. More specifically, in the numerical example, the stable range of h_2 is much larger than that given by the method in [4, 5] when $h_1 < 2.25$; and we can even obtain the stable range of h_2 when $h_1 \geq 2.25$. Note that in this case, the stable range of h_2 no longer starts from 0.

Remark 3: Unlike the discretized-Lyapunov-functional approach, our method can easily be extended to the synthesis of a control system, as explained in [4, 5, 19].

Table 1: Range of h_2 ensuring asymptotic stability of system (21) for a given h_1.

h_1	1.51	1.52	1.53	1.55	1.6
h_2 (Theorem 1)	[0, $+\infty$]	[0, 3.36]	[0, 3.35]	[0, 3.34]	[0, 3.33]
h_2 (Method in $[4,5]$)	[0, $+\infty$]	[0, 1.84]	[0, 1.81]	[0, 1.78]	[0, 1.71]
h_1	1.64	1.7	1.8	1.9	2.0
h_2 (Theorem 1)	[0, 3.33]	[0, 3.33]	[0, 3.36]	[0, 3.39]	[0, 3.43]
h_2 (Method in $[4,5]$)	[0, 1.64]	[0, 1.57]	[0, 1.42]	[0, 1.22]	[0, 0.88]
h_1	2.1	2.2	2.25	2.3	2.4
h_2 (Theorem 1)	[0, 3.47]	[0, 3.52]	[0, 3.55]	[0.08, 3.57]	[0.22, 3.61]
h_2 (Method in $[4,5]$)	[0, 0.40]	[0, 0.07]	[0, 0]	–	–
h_1	2.5	3.0	3.5	4.0	4.47
h_2 (Theorem 1)	[0.35, 3.65]	[1.04, 3.77]	[1.88, 3.90]	[3.59, 4.18]	[4.47, 4.47]
h_2 (Method in $[4,5]$)	–	–	–	–	–

2.3 Multiple Time Delays

In this subsection, Theorem 1 is extended to a system with multiple time delays. For convenience, it is assumed that in (3)

$$0 = h_0 \leq h_1 \leq h_2 \leq \cdots \leq h_m. \tag{23}$$

Theorem 2 *For given scalars $h_i \geq 0$ $(i = 1, 2, \cdots, m)$ that satisfy (23), the system (3) with multiple time delays is asymptotically stable if there exist symmetric positive definite matrices $P = P^T > 0$ and $Q_i = Q_i^T > 0$ $(i = 1, 2, \cdots, m)$, symmetric semi-positive definite matrices* $X^{(ij)} = \begin{bmatrix} X_{00}^{(ij)} & X_{01}^{(ij)} & \cdots & X_{0m}^{(ij)} \\ [X_{01}^{(ij)}]^T & X_{11}^{(ij)} & \cdots & X_{1m}^{(ij)} \\ \vdots & \vdots & \ddots & \vdots \\ [X_{0m}^{(ij)}]^T & [X_{1m}^{(ij)}]^T & \cdots & X_{mm}^{(ij)} \end{bmatrix} \geq 0$ $(0 \leq i < j \leq m)$ *and* $W^{(ij)} = [W^{(ij)}]^T \geq 0$ $(0 \leq i < j \leq m)$, *and any matrices* $N_l^{(ij)}$ $(l = 0, 1, 2, \cdots, m, 0 \leq i < j \leq m)$ *such that the following LMIs hold:*

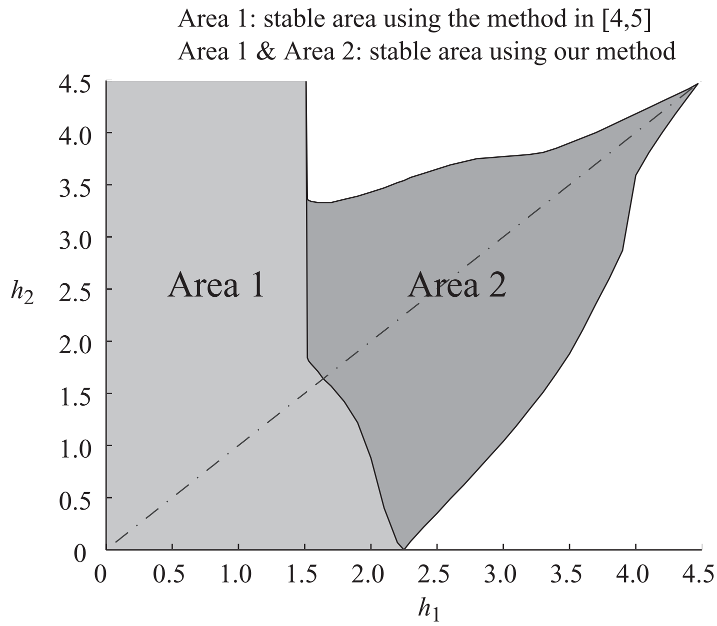

Figure 1: Comparison of stability ranges obtained with the method in [4, 5] and our method.

$$\Xi = \begin{bmatrix} \Xi_{00} & \Xi_{01} & \cdots & \Xi_{0m} \\ \Xi_{01}^T & \Xi_{11} & \cdots & \Xi_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \Xi_{0m}^T & \Xi_{1m}^T & \cdots & \Xi_{mm} \end{bmatrix} < 0, \tag{24}$$

$$\Gamma^{(ij)} = \begin{bmatrix} X_{00}^{(ij)} & X_{01}^{(ij)} & \cdots & X_{0m}^{(ij)} & N_0^{(ij)} \\ [X_{01}^{(ij)}]^T & X_{11}^{(ij)} & \cdots & X_{1m}^{(ij)} & N_1^{(ij)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ [X_{0m}^{(ij)}]^T & [X_{1m}^{(ij)}]^T & \cdots & X_{mm}^{(ij)} & N_m^{(ij)} \\ [N_0^{(ij)}]^T & [N_1^{(ij)}]^T & \cdots & [N_m^{(ij)}]^T & W^{(ij)} \end{bmatrix} \geq 0 \ (0 \leq i < j \leq m), \tag{25}$$

where

$$\begin{aligned} \Xi_{00} &= PA_0 + A_0^T P + \sum_{i=1}^{m} Q_i + \sum_{j=1}^{m} \left(N_0^{(0j)} + [N_0^{(0j)}]^T \right) \\ &\quad + A_0^T G A_0 + \sum_{i=0}^{m} \sum_{j=i+1}^{m} (h_j - h_i) X_{00}^{(ij)}, \\ \Xi_{0k} &= PA_k - \sum_{i=0}^{k-1} N_0^{(ik)} + \sum_{i=1}^{m} [N_k^{(0i)}]^T + \sum_{j=k+1}^{m} N_0^{(kj)} \end{aligned}$$

$$
\begin{aligned}
& +A_0^T G A_k + \sum_{i=0}^{m}\sum_{j=i+1}^{m}(h_j-h_i)X_{0k}^{(ij)} \ (k=1,2,\cdots,m),\\
\Xi_{kk} \ &= \ -Q_k - \sum_{i=0}^{k-1}\left(N_k^{(ik)} + [N_k^{(ik)}]^T\right) + \sum_{j=k+1}^{m}\left(N_k^{(kj)} + [N_k^{(kj)}]^T\right)\\
& +A_k^T G A_k + \sum_{i=0}^{m}\sum_{j=i+1}^{m}(h_j-h_i)X_{kk}^{(ij)} \ (k=1,2,\cdots,m),\\
\Xi_{lk} \ &= \ -\sum_{i=0}^{k-1} N_l^{(ik)} - \sum_{i=0}^{l-1}[N_k^{(il)}]^T + \sum_{j=k+1}^{m} N_l^{(kj)} + \sum_{j=l+1}^{m}[N_k^{(lj)}]^T\\
& +A_l^T G A_k + \sum_{i=0}^{m}\sum_{j=i+1}^{m}(h_j-h_i)X_{lk}^{(ij)} \ (l=1,2,\cdots,m, l<k\le m),\\
G \ &= \ \sum_{i=0}^{m}\sum_{j=i+1}^{m}(h_j-h_i)W^{(ij)}.
\end{aligned}
$$

Proof. Choose the candidate Lyapunov-Krasovskii functional to be

$$
\begin{aligned}
V_m(x_t) \ &:= \ x^T(t)^T P x(t) + \sum_{i=1}^{m}\int_{t-h_i}^{t} x^T(s)Q_i x(s)ds\\
& + \sum_{i=0}^{m-1}\sum_{j=i+1}^{m}\int_{-h_j}^{-h_i}\int_{t+\theta}^{t}\dot{x}^T(s)W^{(ij)}\dot{x}(s)dsd\theta,
\end{aligned} \tag{26}
$$

where $P=P^T>0$, $Q_i=Q_i^T>0$ $(i=1,2,\cdots,m)$, and $W^{(ij)}=[W^{(ij)}]^T\ge 0$ $(0\le i<j\le m)$ are free matrices that need to be determined. According to the Leibniz-Newton formula, for $0\le i<j\le m$ and for any matrices $N_l^{(ij)}$ $(l=0,1,2,\cdots,m)$, the following equation holds:

$$
2\left[\sum_{l=0}^{m} x^T(t-h_l)N_l^{(ij)}\right]\cdot\left[x(t-h_i)-x(t-h_j)-\int_{t-h_j}^{t-h_i}\dot{x}(s)ds\right]=0. \tag{27}
$$

On the other hand, for $0\le i<j\le m$ and for any matrices $X^{(ij)}\ge 0$,

$$
\sum_{i=0}^{m-1}\sum_{j=i+1}^{m}(h_j-h_i)\zeta_1^T(t)[X^{(ij)}-X^{(ij)}]\zeta_1(t)=0, \tag{28}
$$

where

$$
\zeta_1(t)=[x^T(t) \ \ x^T(t-h_1) \ \ x^T(t-h_2) \ \ \cdots \ \ x^T(t-h_m)]^T.
$$

So, the derivative of $V_m(x_t)$ along the solutions of Σ can be written as

$$
\dot{V}_m(x_t)=\zeta_1^T(t)\Xi\zeta_1(t)-\sum_{i=0}^{m-1}\sum_{j=i+1}^{m}\int_{t-h_j}^{t-h_i}\zeta_2^T(t,s)\Gamma^{(ij)}\zeta_2(t,s)ds, \tag{29}
$$

where

$\zeta_2(t,s) = [\zeta_1^T(t) \;\; \dot{x}(s)]^T,$

and Ξ and $\Gamma^{(ij)}$ $(0 \le i < j \le m)$ are defined in (24) and (25). From (23), Σ is asymptotically stable if LMIs (24) and (25) hold. ∎

Remark 5: If $\exists i \in [1, \cdots, m-1]$ such that $h_i = h_{i+1}$, then the system can be transformed into a system with $m-1$ time delays. Following the explanation in Theorem 1 and Corollary 1, it is easy to see that the delay-dependent condition is equivalent to that for a system with $m-1$ time delays.

3 Conclusion

This paper presented some new delay-dependent stability criteria for linear systems with multiple delays. The free-weighting-matrix method was used in the derivation. Since the method does not employ a system transformation, does not use an inequality to estimate the upper bound on a cross term, and uses free matrices to take the relationships between h_i $(i = 0, 1, 2, \cdots, m)$ into account, these criteria overcome the conservatism of previous methods. Free weighting matrices that express the reciprocal influences of the terms in the Leibniz-Newton formula can easily be calculated and are determined by LMIs. Unlike existing methods, the stability domain consists of ranges for the time delays rather than just upper bounds. A numerical example demonstrated that the method described in this paper is a significant improvement over previous methods.

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