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On Absolute Stability of Lur'e Control Systems with Multiple Nonlinearities

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Abstract

This paper presents necessary and sufficient conditions for the existence of a Lyapunov function in the Lur'e form that guarantees the absolute stability of a Lur'e control system with multiple nonlinearities. It simplifies the existence problem to one of solving a set of linear matrix inequalities (LMIs). If these inequalities are feasible, the free parameters in the Lyapunov function, such as the positive definite matrix and the coefficients of the integral terms, are given by the solution of the LMIs. Otherwise, there does not exist any Lyapunov function in the Lur'e form. Necessary and sufficient conditions are also obtained for the robust absolute stability of time-varying structured uncertain systems.

Keywords: Lur'e control systems, absolute stability, robustness, Lyapunov function, linear matrix inequality (LMI).

1 Introduction

The absolute stability of Lur'e control systems has been discussed by many researchers. Most of their results were obtained by using the Popov frequency domain criteria (e.g., [4, 8, 11, 13]), the extended Popov frequency domain criteria (e.g., [12, 14]), or the Lyapunov function in the Lur'e form (e.g., [5, 20]). It is difficult to deal with systems with multiple nonlinearities using the Popov criteria since the criteria are not geometrically intuitive and cannot be examined by

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illustration; and the extended Popov frequency domain criteria in [12] are only sufficient conditions for systems with multiple nonlinearities. For the method employing a Lyapunov function in the Lur'e form, the necessary and sufficient conditions for the existence of a Lyapunov function that have been obtained so far are for Lur'e control systems with multiple nonlinearities in a bounded sector [5, 20]. However, they are only existence conditions, and are not solvable. Furthermore, the criteria for examining absolute stability depend on the selection of free parameters, such as a positive definite matrix and the coefficients of integral terms. Since those parameters cannot be obtained by analytical or numerical methods, the criteria are not very practical. For example, the fact that the appropriate parameters cannot be found does not necessarily mean that there does not exist a Lyapunov function in the Lur'e form that guarantees the absolute stability of the system.

Sufficient conditions for the existence of a Lyapunov function in the Lur'e form that guarantees the absolute stability of Lur'e control systems have been found by using a linear matrix inequality (LMI) and the S-procedure [1]. Moreover, the free parameters, such as the positive definite matrix and the coefficients of the integral terms of the Lyapunov function, can be obtained from the solution of the LMI. These conditions are also necessary if there is only a single nonlinearity in the system. Unfortunately, they are only sufficient conditions if there is more than one nonlinearity due to the losslessness of the S-procedure [9].

For the robust absolute stability of various uncertain Lur'e control systems, many sufficient conditions have been derived based on the Popov frequency domain criteria and a method employing a Lyapunov function in the Lur'e form [2, 6, 7, 10, 15–18]. Regarding the issue of robust absolute stability, the same difficulty is encountered for uncertain systems with multiple nonlinearities as for certain systems. And there are few results on the robust absolute stability of Lur'e control systems with time-varying structured uncertainties.

This paper discusses the problem of the existence of a Lyapunov function in the Lur'e form that guarantees the absolute stability of Lur'e control systems with multiple nonlinearities in a bounded sector. The problem is converted to one of solving a set of LMIs, and necessary and sufficient conditions for the existence problem are presented. It is shown that there exists a Lyapunov function in the Lur'e form that guarantees absolute stability if those LMIs are feasible. Moreover, the free parameters, such as the positive definite matrix and the coefficients of the integral terms of the Lyapunov function, are given by the solution of the LMIs. And if the LMIs are false, no such Lyapunov function exists. Furthermore, the results obtained are extended to Lur'e control systems with time-varying structured uncertainties; and necessary and sufficient conditions for their robust absolute stability are obtained by using the necessary and sufficient conditions given in [19]. The LMIs can be solved using the Matlab LMI toolbox [3].

This paper is organized as follows: Section 2 provides preliminary information needed in the rest of the paper. Section 3 presents necessary and sufficient conditions for the absolute stability of Lur'e control systems. Section 4 discusses robust absolute stability criteria for uncertain systems. Section 5 presents some numerical examples that illustrate the effectiveness of the proposed method. And some conclusions are drawn in Section 6.

2 Preliminaries

Consider a Lur'e control system with multiple nonlinearities and uncertainties.

$$\mathcal{S}: \begin{cases} \dot{x} = (A + \triangle A(t))x + (B + \triangle B(t))f(\sigma), \\ \sigma = C^T x, \end{cases} \tag{1}$$

where $x \in R^n$ is the state; $f(\sigma) = [f_1(\sigma_1)\ f_2(\sigma_2)\ \cdots\ f_m(\sigma_m)]^T \in R^{m\times 1}$ is a nonlinear function; $A \in R^{n\times n}$, $B = [b_1\ b_2\ \cdots\ b_m] \in R^{n\times m}$, $C = [c_1\ c_2\ \cdots\ c_m] \in R^{n\times m}$ and $c_j\ (j = 1, 2, \cdots, m)$ are linear-independent; and $\sigma = [\sigma_1\ \sigma_2\ \cdots\ \sigma_m]^T \in R^{m\times 1}$.

The nominal system of $\mathcal{S}$ is given by

$$\mathcal{S}_0: \begin{cases} \dot{x} = Ax + Bf(\sigma), \\ \sigma = C^T x. \end{cases} \tag{2}$$

In (1) and (2), the nonlinearities $f_j(\sigma_j)$ satisfy

$$f_j(\sigma_j) \in K_j[0, k_j] = \{f_j(\sigma_j) |\, f_j(0) = 0, \quad \text{and} \quad 0 \leq \sigma_j f_j(\sigma_j) \leq k_j \sigma_j^2 \quad \text{for} \quad \sigma_j \neq 0\,\} \tag{3}$$

for $0 < k_j < +\infty$, $j = 1, 2, \cdots, m$.

The uncertainties are assumed to be of the following form:

$$[\Delta A(t)\ \Delta B(t)] = DF(t)\,[E_a\ E_b]\,, \tag{4}$$

where D, E_a and E_b are known real constant matrices with appropriate dimensions; and $F(t)$ is an unknown real, time-varying, appropriately dimensioned matrix with Lebesgue-measurable elements satisfying

$$\|F(t)\| \leq 1,\ \forall t, \tag{5}$$

where $\|\cdot\|$ is the Euclidean norm. For simplicity,

$$\bar{A} = A + \Delta A(t) \text{ and } \bar{B} = B + \Delta B(t) \tag{6}$$

are used in the rest of this paper.

Definition 1 *Assume that the sector is bounded by*

$$K = \operatorname{diag}(\, k_1, \quad k_2, \quad \cdots, \quad k_m\,). \tag{7}$$

The system $\mathcal{S}$ (or the nominal system $\mathcal{S}_0$) is said to be robustly absolutely stable (or absolutely stable) in the sector if $\mathcal{S}$ (or $\mathcal{S}_0$) is globally robustly stable (or globally asymptotically stable) for $f_j(\sigma_j) \in K_j[0, k_j]$ $(j = 1, 2, \cdots, m)$.

Construct the following Lyapunov function in the Lur'e form:

$$V(x) = x^T P x + 2 \sum_{j=1}^{m} \lambda_j \int_0^{\sigma_j} f_j(\sigma_j) d\sigma_j, \tag{8}$$

where $P = P^T > 0$ and $\lambda_j \geq 0$ $(j = 1, 2, \cdots, m)$ need to be determined. If the function $V(x)$ in (8) satisfies

$$\dot{V}_{\mathcal{S}} < 0 \quad (\text{or } \dot{V}_{\mathcal{S}_0} < 0), \quad \text{for} \quad x \neq 0 \quad \text{and any} \quad f_j(\sigma_j) \in K_j[0, k_j], \quad j = 1, 2, \cdots, m, \tag{9}$$

where $\dot{V}_{\mathcal{S}}$ (or $\dot{V}_{\mathcal{S}_0}$) is the derivative of the Lyapunov function with respect to time for $\mathcal{S}$ (or $\mathcal{S}_0$), then $\mathcal{S}$ (or $\mathcal{S}_0$) is robustly absolutely stable (or absolutely stable) in the sector bounded by (7). This is stated in the following lemma.

Lemma 1 *[5,20] For a nominal system $\mathcal{S}_0$ with $m \geq 2$, the necessary and sufficient conditions for inequality (9) to hold are*

(a) $\dot{V}_{\mathcal{S}_0} < 0$ for $x \neq 0$, $f_1(\sigma_1) = \alpha_1 \sigma_1$ $(\alpha_1 \in \{0, k_1\})$, and $f_j(\sigma_j) \in K_j[0, k_j]$ $(j = 2, 3, \cdots, m)$; and

(b) $\dot{V}_{\mathcal{S}_0} < 0$ for $x \neq 0$, $f_1(\sigma_1) \in K_1[0, k_1]$, and $f_j(\sigma_j) = 0$ $(j = 2, 3, \cdots, m)$.

Proof. See the appendix.

From the procedure for the proof of Lemma 1, it can be seen that the conclusion is also true when A and B are time-varying. So, Lemma 1 holds for the uncertain system $\mathcal{S}$ as well.

The S-procedure [1,9] plays a very important role in this study. It is given in the following lemma.

Lemma 2 *[1, 9] (S-procedure) Let $T_i \in R^{n\times n}$ $(i = 0, 1, \cdots, p)$ be symmetric matrices. The conditions on T_i $(i = 0, 1, \cdots, p)$,*

$$\zeta^T T_0 \zeta > 0 \ \ \textit{for all } \zeta \neq 0 \ \ \textit{such that } \zeta^T T_i \zeta \geq 0 \ (i = 1, 2, \cdots, p), \tag{10}$$

hold if there exist $\tau_i \geq 0$ $(i = 1, 2, \cdots, p)$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0. \tag{11}$$

When $p = 1$, the existence of a ζ_0 such that $\zeta_0^T T_1 \zeta_0 > 0$ is also a necessary condition.

Boyd *et al.* [1] used the S-procedure to derive a sufficient condition (Note that it is also necessary when $p = 1$.). The following lemma presents a different version of this condition.

Lemma 3 *Inequality (9) holds for the nominal system $\mathcal{S}_0$ if there exist $P = P^T > 0$, $T = diag(t_1, t_2, \cdots, t_m) \geq 0$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_m) \geq 0$ such that the LMI*

$$\Phi = \begin{bmatrix} A^T P + PA & PB + A^T C\Lambda + CKT \\ B^T P + \Lambda C^T A + TKC^T & \Lambda C^T B + B^T C\Lambda - 2T \end{bmatrix} < 0 \tag{12}$$

is feasible, where K is given by (7). This is also a necessary condition when $m = 1$.

To obtain necessary and sufficient conditions for a system with time-varying structured uncertainties, the following lemma is needed to deal with the uncertainties.

Lemma 4 *[19] For given matrices $Q = Q^T, H, E$ and $R = R^T > 0$ of appropriate dimensions,*

$$Q + HF(t)E + E^T F^T(t) H^T < 0$$

holds for all $F(t)$ satisfying $F^T(t)F(t) \leq R$, if and only if there exists some $\varepsilon > 0$ such that

$$Q + \varepsilon H H^T + \varepsilon^{-1} E^T R E < 0.$$

3 Absolute Stability

Lemma 3 gives a sufficient condition for the existence of a Lyapunov function in the Lur'e form that guarantees the absolute stability of Lur'e control systems with multiple nonlinearities. Since the S-procedure is directly employed in this lemma, it is generally conservative. This section presents more practical necessary and sufficient conditions.

Let

$$\Gamma_{1\sim m} = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_m).$$

Then,

$$D_j^{1\sim m} = \{\Gamma_{1\sim m} | \alpha_i = 0 \ for \ i \geq j, \ \alpha_i \in \{0, k_i\} \ for \ i < j, \ (i = 1, 2, \cdots, m)\}, \ j = 1, 2, \cdots, m \tag{13}$$

contains 2^{j-1} elements. First, we have the following theorem for a system without uncertainties.

Theorem 1 *For the nominal system $\mathcal{S}_0$ with $m \geq 1$, the necessary and sufficient condition for (9) is that $\dot{V}_{\mathcal{S}_0} < 0$ holds for all $j = 1, 2, \cdots, m$ and $\Gamma_{1\sim m} \in D_j^{1\sim m}$ when $x \neq 0$, $f_i(\sigma_i) = \alpha_i \sigma_i$ $(i = 1, 2, \cdots, m, \ i \neq j)$, and $f_j(\sigma_j) \in K_j[0, k_j]$.*

Proof. The inductive method is used to prove this theorem. From Lemma 1, the theorem holds for $m = 1$. Suppose that it holds for $m = \rho$. Now, consider the system with ρ nonlinearities

$$\dot{x} = Ax + \sum_{j=2}^{\rho+1} b_j f_j(\sigma_j). \tag{14}$$

Let $D_j^{2\sim(\rho+1)} = \{\Gamma_{2\sim(\rho+1)} | \alpha_i = 0 \text{ for } i \geq j, \alpha_i \in \{0, k_i\} \text{ for } i < j, \ (i = 2, 3, \cdots, \rho+1)\}$ $(j = 2, 3, \cdots, \rho+1)$. The necessary and sufficient condition for (9) is that $\dot{V}_{\mathcal{S}_0} < 0$ holds for all $j = 2, 3, \cdots, \rho+1$ and $\Gamma_{2\sim(\rho+1)} \in D_j^{2\sim(\rho+1)}$ when $x \neq 0$, $f_i(\sigma_i) = \alpha_i \sigma_i$ $(i = 2, 3, \cdots, \rho+1$, $i \neq j)$, and $f_j(\sigma_j) \in K_j[0, k_j]$.

(9) holds if and only if conditions (a) and (b) in Lemma 1 hold for $m = \rho+1$. The necessary and sufficient condition for condition (b) is that $\dot{V}_{\mathcal{S}_0} < 0$ holds for any $\Gamma_{1\sim(\rho+1)} \in D_1^{1\sim(\rho+1)} = \{\mathrm{diag}\{0, 0, \cdots, 0\}\}$ when $x \neq 0$, $f_i(\sigma_i) = \alpha_i \sigma_i$ $(i = 2, 3, \cdots, \rho+1)$, and $f_1(\sigma_1) \in K_1[0, k_1]$.

The necessary and sufficient conditions for condition (a) can be divided into two cases:

(i) If $\alpha_1 = 0$ and $f_1(\sigma_1) = 0$, then $\mathcal{S}_0$ is given by Eq. (14). Let

$$\bar{D}_j^{1\sim(\rho+1)} = \{\Gamma_{1\sim(\rho+1))} | \alpha_1 = 0, \ \Gamma_{2\sim(\rho+1)} \in D_j^{2\sim(\rho+1)}\} \ (j = 2, 3, \cdots, \rho+1).$$

The necessary and sufficient condition for (9) is that $\dot{V}_{\mathcal{S}_0} < 0$ holds for all $j = 2, 3, \cdots, \rho+1$ and $\Gamma_{1\sim(\rho+1)} \in \bar{D}_j^{1\sim(\rho+1)}$ when $x \neq 0$, $f_i(\sigma_i) = \alpha_i \sigma_i$ $(i = 1, 2, \cdots, \rho+1, i \neq j)$, and $f_j(\sigma_j) \in K_j[0, k_j]$.

(ii) If $\alpha_1 = k_1$ and $f_1(\sigma_1) = k_1 \sigma_1$, then $\mathcal{S}_0$ can be described by

$$\dot{x} = Ax + k_1 b_1 \sigma_1 + \sum_{j=2}^{\rho+1} b_j f_j(\sigma_j). \tag{15}$$

Let

$$\hat{D}_j^{1\sim(\rho+1)} = \left\{\Gamma_{1\sim(\rho+1)} | \, \alpha_1 = k_1, \quad \Gamma_{2\sim(\rho+1)} \in D_j^{2\sim(\rho+1)}\right\}; \; j = 2, 3, \cdots, \rho+1.$$

The necessary and sufficient condition for (9) is that $\dot{V}_{\mathcal{S}_0} < 0$ holds for all $j = 2, 3, \cdots, \rho+1$ and $\Gamma_{1\sim(\rho+1)} \in \hat{D}_j^{1\sim(\rho+1)}$ when $x \neq 0$, $f_i(\sigma_i) = \alpha_i\sigma_i$ $(i = 1, 2, \cdots, \rho+1, i \neq j)$, and $f_j(\sigma_j) \in K_j[0, k_j]$.

As a result,

$$D_j^{1\sim(\rho+1)} = \begin{cases} D_1^{1\sim(\rho+1)}, & j = 1; \\ \bar{D}_j^{1\sim(\rho+1)} \bigcup \hat{D}_j^{1\sim(\rho+1)}, & j = 2, 3, \cdots, \rho+1. \end{cases} \tag{16}$$

So, conditions (a) and (b) in Lemma 1 are equivalent to the case where $\dot{V}_{\mathcal{S}_0} < 0$ holds for all $j = 1, 2, \cdots, \rho+1$ and $\Gamma_{1\sim(\rho+1)} \in D_j^{1\sim(\rho+1)}$ when $x \neq 0$, $f_i(\sigma_i) = \alpha_i\sigma_i$ $(i = 1, 2, \cdots, \rho+1$, $i \neq j)$, and $f_j(\sigma_j) \in K_j[0, k_j]$. Thus, the theorem also holds for $m = \rho + 1$. ∎

Theorem 1 gives the necessary and sufficient conditions for (9) by converting multiple nonlinearities into a simple form with only a single nonlinearity. The following theorem presents the conditions in the form of an LMI. For simplicity, $\Gamma_{1\sim m}$ is abbreviated as Γ hereafter; and we assume that

$$A(\Gamma) := A + B\Gamma C^T \text{ and } P(\Gamma) := P + C\Lambda\Gamma C^T. \tag{17}$$

Theorem 2 *The necessary and sufficient conditions for the existence of the Lyapunov function $V(x)$ in (8) satisfying (9) that ensures that the system $\mathcal{S}_0$ is absolutely stable in the sector bounded by (7) are that there exist $P = P^T > 0$, $t_\Gamma \geq 0$ and $\lambda_j \geq 0$ such that the following LMI is feasible for all $\Gamma \in D_j^{1\sim m}$ $(j = 1, 2, \cdots, m)$:*

$$H_j(\Gamma) = \begin{bmatrix} A^T(\Gamma)P(\Gamma) + P(\Gamma)A(\Gamma) & P(\Gamma)b_j + \lambda_j A^T(\Gamma)c_j + t_\Gamma k_j c_j \\ b_j^T P(\Gamma) + \lambda_j c_j^T A(\Gamma) + t_\Gamma k_j c_j^T & 2\lambda_j c_j^T b_j - 2t_\Gamma \end{bmatrix} < 0. \tag{18}$$

Proof. Consider the case where $\Gamma \in D_j^{1\sim m}$, $f_i(\sigma_i) = \alpha_i\sigma_i$ and $f_j(\sigma_j) \in K_j[0, k_j]$ $(i, j = 1, 2, \cdots, m, i \neq j)$. $\mathcal{S}_0$ is given by

$$\begin{aligned} \tilde{\mathcal{S}}: \; \dot{x} &= Ax + \sum_{i=1}^{m} b_i f_i(\sigma_i) = Ax + \sum_{\substack{i=1 \\ i\neq j}}^{m} b_i\alpha_i\sigma_i + b_j f_j(\sigma_j) \\ &= Ax + \sum_{\substack{i=1 \\ i\neq j}}^{m} b_i\alpha_i c_i^T x + b_j f_j(\sigma_j) = A(\Gamma)x + b_j f_j(\sigma_j), \end{aligned} \tag{19}$$

and the Lyapunov function is

$$
\begin{aligned}
V(x) &= x^T P x + 2\sum_{\substack{i=1\\ i\neq j}}^{m} \lambda_i \int_0^{\sigma_i} \alpha_i \sigma_i d\sigma_i + 2\lambda_j \int_0^{\sigma_j} f_j(\sigma_j) d\sigma_j \\
&= x^T P x + \sum_{\substack{i=1\\ i\neq j}}^{m} \lambda_i \alpha_i \sigma_i^2 + 2\lambda_j \int_0^{\sigma_j} f_j(\sigma_j) d\sigma_j \\
&= x^T P x + \sum_{\substack{i=1\\ i\neq j}}^{m} \lambda_i \alpha_i x^T c_i c_i^T x + 2\lambda_j \int_0^{\sigma_j} f_j(\sigma_j) d\sigma_j \\
&= x^T P(\Gamma) x + 2\lambda_j \int_0^{\sigma_j} f_j(\sigma_j) d\sigma_j.
\end{aligned} \tag{20}
$$

To calculate the derivative of $V(x)$ for Eq. (19), the inequality

$$
\dot{V}_{\tilde{S}} = \begin{bmatrix} x^T & f_j^T(\sigma_j) \end{bmatrix} \begin{bmatrix} A^T(\Gamma)P(\Gamma) + P(\Gamma)A(\Gamma) & P(\Gamma)b_j + \lambda_j A^T(\Gamma)c_j \\ b_j^T P(\Gamma) + \lambda_j c_j^T A(\Gamma) & 2\lambda_j c_j^T b_j \end{bmatrix} \begin{bmatrix} x \\ f_j(\sigma_j) \end{bmatrix} < 0 \tag{21}
$$

must be true for $x \neq 0$ under the condition (3). On the other hand, condition (3), $f_j(\sigma_j) \in K_j[0, k_j]$ $(j = 1, 2, \cdots, m)$, is equivalent to

$$
f_j(\sigma_j)(f_j(\sigma_j) - k_j c_j^T x)) \leq 0. \tag{22}
$$

This yields

$$
\begin{aligned}
\{(x, f_j(\sigma_j))| x \neq 0, f_j(\sigma_j) \in K_j[0, k_j]\} = \{(x, f_j(\sigma_j))| x \neq 0 \text{ or } f_j(\sigma_j) \neq 0,\ f_j(\sigma_j) \in K_j[0, k_j]\}, \\
j = 1, 2, \cdots, m.
\end{aligned} \tag{23}
$$

Since there is only a single nonlinear term in Eq. (19), it is clear from Lemma 2 (S-procedure) that the necessary and sufficient conditions for (21) are that there exists $t_\Gamma \geq 0$ such that LMI (18) is feasible. ∎

Remark 1 *[5, 20] presented only existence conditions that are not solvable. The criteria for absolute stability depend on the selection of free parameters, such as the positive definite matrix and the coefficients of the integral terms. However, it is not easy to determine the parameters by existing methods. In contrast, the free parameters in Theorem 2 can easily be obtained by solving LMI (18). So, Theorem 2 is more practical.*

Remark 2 *Although the free parameters in the criteria in [1] can be derived from the solution of an LMI, they are only sufficient conditions for the existence of a Lyapunov function in the Lur'e form that guarantees the absolute stability of a Lur'e control system with multiple nonlinearities. On the other hand, the conditions in Theorem 2 are necessary, even when there is more than one nonlinearity.*

4 Robustness for Absolute Stability

For a time-varying structured uncertain system, $\mathcal{S}$, the following sufficient condition is derived by directly employing the S-procedure. The uncertainties are dealt with by using Lemma 4.

Theorem 3 *System $\mathcal{S}$ is robustly absolutely stable in a sector bounded by $K = diag(k_1, k_2, \cdots, k_m)$ if there exist $P = P^T > 0$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_m) \geq 0$, $T = diag(t_1, t_2, \cdots, t_m) \geq 0$ and $\varepsilon > 0$ such that the following LMI is feasible:*

$$\begin{bmatrix} A^TP + PA + \varepsilon E_a^T E_a & PB + A^T C\Lambda + CKT + \varepsilon E_a^T E_b & PD \\ B^TP + \Lambda C^T A + TKC^T + \varepsilon E_b^T E_a & \Lambda C^T B + B^T C\Lambda - 2T + \varepsilon E_b^T E_b & \Lambda C^T D \\ D^T P & D^T C\Lambda & -\varepsilon I \end{bmatrix} < 0. \quad (24)$$

Proof. If we replace A and B in (12) with $A + DF(t)E_a$ and $B + DF(t)E_b$, respectively, then (12) for $\mathcal{S}$ is equivalent to the following condition:

$$\Phi + \begin{bmatrix} PD \\ \Lambda C^T D \end{bmatrix} F(t) \begin{bmatrix} E_a & E_b \end{bmatrix} + \begin{bmatrix} E_a^T \\ E_b^T \end{bmatrix} F^T(t) \begin{bmatrix} D^T P & D^T C\Lambda \end{bmatrix} < 0. \quad (25)$$

By Lemma 4, a necessary and sufficient condition guaranteeing (12) for $\mathcal{S}$ is that there exists a positive number $\varepsilon > 0$ such that

$$\Phi + \varepsilon^{-1} \begin{bmatrix} PD \\ \Lambda C^T D \end{bmatrix} \begin{bmatrix} D^T P & D^T C\Lambda \end{bmatrix} + \varepsilon \begin{bmatrix} E_a^T \\ E_b^T \end{bmatrix} \begin{bmatrix} E_a & E_b \end{bmatrix} < 0. \quad (26)$$

Applying the Schur complement shows that (26) is equivalent to (24). ∎

The conditions in Theorem 3 for the robust absolute stability of system $\mathcal{S}$ with multiple nonlinearities are conservative because they are only sufficient conditions. To reduce the conservatism, the following theorem is derived based on the necessary and sufficient conditions obtained in the previous section.

Theorem 4 *The necessary and sufficient conditions for the existence of the Lyapunov function $V(x)$ in (8) satisfying (9) that ensures that the system $\mathcal{S}$ is robustly absolutely stable in the sector bounded by (7) is that, for all $j = 1, 2, \cdots, m$ and $\Gamma \in D_j^{1\sim m}$, there exist $P = P^T > 0$, $\lambda_j \geq 0$, $t_\Gamma \geq 0$ and $\varepsilon_\Gamma \geq 0$ such that the following LMI is feasible:*

$$\begin{bmatrix} A^T(\Gamma)P(\Gamma) + P(\Gamma)A(\Gamma) + \varepsilon_\Gamma E_a^T(\Gamma)E_a(\Gamma) & \Psi_{12} + \varepsilon_\Gamma E_a^T(\Gamma)E_{bj} & P(\Gamma)D \\ \Psi_{12}^T + \varepsilon_\Gamma E_{bj}^T E_a(\Gamma) & 2\lambda_j c_j^T b_j + \varepsilon_\Gamma E_{bj}^T E_{bj} - 2t_\Gamma & \lambda_j c_j^T D \\ D^T P(\Gamma) & \lambda_j D^T c_j & -\varepsilon_\Gamma I \end{bmatrix} < 0, \quad (27)$$

where $E_a(\Gamma) := (E_a + E_b\Gamma C^T)^T(E_a + E_b\Gamma C^T)$ and $\Psi_{12} = P(\Gamma)b_j + \lambda_j A^T(\Gamma)c_j + t_\Gamma k_j c_j$.

Proof. Let $\bar{b}_j$ be the j-th column of $\bar{B}$. From Theorem 2, the conditions in (18) for system $\mathcal{S}$ are equivalent to the condition that there exist $P = P^T > 0$, $\lambda_j \geq 0$ and $t_\Gamma \geq 0$, for all $j = 1, 2, \cdots, m$ and $\Gamma \in D_j^{1\sim m}$ such that the following holds:

$$\bar{H}_j(\Gamma) = \begin{bmatrix} \bar{A}^T(\Gamma)P(\Gamma) + P(\Gamma)\bar{A}(\Gamma) & P(\Gamma)\bar{b}_j + \lambda_j \bar{A}^T(\Gamma)c_j + t_\alpha k_j c_j \\ \bar{b}_j^T P(\Gamma) + \lambda_j c_j^T \bar{A}(\Gamma) + t_\Gamma k_j c_j^T & 2\lambda_j c_j^T \bar{b}_j - 2t_\Gamma \end{bmatrix} < 0. \tag{28}$$

Replacing $\bar{A}(\Gamma)$ and $\bar{b}_j$ in (28) with $A(\Gamma) + DF(t)E_a(\Gamma)$ and $b_j + DF(t)E_{bj}$, respectively, allows us to rewrite $\bar{H}_j(\Gamma)$ as

$$\bar{H}_j(\Gamma) = H_j(\Gamma) + \begin{bmatrix} P(\Gamma)D \\ \lambda_j c_j^T D \end{bmatrix} F(t) \begin{bmatrix} E_a(\Gamma) & E_{bj} \end{bmatrix} + \begin{bmatrix} E_a^T(\Gamma) \\ E_{bj}^T \end{bmatrix} F^T(t) \begin{bmatrix} D^T P(\Gamma) & \lambda_j D^T c_j \end{bmatrix}, \tag{29}$$

where $H_j(\Gamma)$ is defined in (18). From Lemma 4 and the Schur complement, $\bar{H}_j(\Gamma) < 0$ if and only if LMI (27) is feasible. ∎

5 Examples

Example 1: Consider the nominal system $\mathcal{S}_0$ with

$$A = \begin{bmatrix} -2 & 0 \\ 2 & -4 \end{bmatrix}, \; B = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{30}$$

and $k_1 = 1$ and $k_2 = 10$.

Since $m = 2$,

$$\begin{aligned} D_1^{1\sim 2} &= \{\mathrm{diag}(0,0)\}, \\ D_2^{1\sim 2} &= \{\mathrm{diag}(0,0),\ \mathrm{diag}(k_1,0)\}. \end{aligned} \tag{31}$$

Solving LMI (18) yields the following parameters for the Lyapunov function:

$$P = \begin{bmatrix} 0.3863 & 0.0423 \\ 0.0423 & 0.1247 \end{bmatrix}, \; \lambda_1 = 0.0174, \; \lambda_2 = 0.7006.$$

So, $\mathcal{S}_0$ is absolutely stable.

If the sector of one nonlinearity is fixed by setting $k_1 = 1$, then the maximum sector bound on the other nonlinearity that guarantees the absolutely stability of $\mathcal{S}_0$ is found to be $k_2 = 17.48$ by solving LMI (18). Specifically, solving LMI (18) yields

$$P = \begin{bmatrix} 0.5973 & -0.5932 \\ -0.5932 & 1.0959 \end{bmatrix} \times 10^4, \; \lambda_1 = 0.8003, \; \lambda_2 = 1.1324 \times 10^4.$$

On the other hand, LMI (12) in Lemma 3 no longer holds for $k_1 = 1$ and $k_2 = 16$. This demonstrates the conservatism of Lemma 3, which deals with absolute stability by applying the S-procedure directly to a system with multiple nonlinearities. In addition, the maximum bound on k_1 can also be obtained by fixing k_2 in the same manner.

Now let's consider an example of the uncertain system $\mathcal{S}$.

Example 2: The system matrices A, B and C are the same as those in (30). The uncertainties $\Delta A(t)$ and $\Delta B(t)$ are

$$\|\Delta A(t)\| \leq 0.5 \text{ and } \|\Delta B(t)\| \leq 0.05.$$

And $k_1 = 1$ and $k_2 = 6.46$.

These uncertainties can be represented in the form of (4) with

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; E_a = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \; E_b = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}.$$

Solving LMI (27) yields

$$P = \begin{bmatrix} 0.5113 & -0.4547 \\ -0.4547 & 1.2904 \end{bmatrix} \times 10^4, \; \lambda_1 = 4, \; \lambda_2 = 8.5467 \times 10^3.$$

So, $\mathcal{S}$ is robustly absolutely stable. At the same time, LMI (24) in Theorem 3 is found to be not feasible for $k_1 = 1$ and $k_2 = 6.14$. This also demonstrates that it is conservative to use the S-procedure to deal directly with multiple nonlinearities in uncertain systems.

6 Conclusions

This paper presents the necessary and sufficient conditions for the existence of a Lyapunov function in the Lur'e form that guarantees the absolute stability of Lur'e control systems with multiple nonlinearities. The existence problem is converted to a simple one of solving a set of LMIs. It was shown that, if the LMIs are feasible, the positive definite matrix and the coefficients of the integral terms of the Lyapunov function are given by the solution of those LMIs. And a Lyapunov function does not exist if the LMIs are not feasible. Furthermore, it was shown that the maximum bounded sector can be found if a Lyapunov function in the Lur'e form exists, which ensures the absolute stability of the system. Finally, less conservative necessary and sufficient conditions were obtained for the robust absolute stability of uncertain systems.

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Appendix A. Proof of Lemma 1

This appendix gives the proof of Lemma 1, which was originally shown in [20].

A simple calculation yields

$$-\dot{V}_{\mathcal{S}_0} = \begin{pmatrix} \xi \\ x \end{pmatrix}^T \begin{pmatrix} -W & -U^T \\ -U & G \end{pmatrix} \begin{pmatrix} \xi \\ x \end{pmatrix}, \tag{32}$$

where

$$\xi = \mathrm{col}(f_1(\sigma_1), \cdots, f_m(\sigma_m)),$$

$$W = \frac{1}{2}(\Lambda C^T B + B^T C \Lambda),$$

$$G = -(A^T P + PA),$$

$$U = (u_1,\ u_2,\ \cdots,\ u_m),\ u_j = Pb_j + \frac{1}{2}\lambda_j A^T c_j,\ j = 1, 2, \cdots, m.$$

Without loss of generality, in the proof of Lemma 1 we can assume $\sigma_j = x_j$. In fact, since c_j $(j = 1, 2, \cdots, m)$ are linear-independent, the system $\mathcal{S}_0$ can be transformed into the above form by means of a nonsingular linear transformation.

Letting $\bar{\sigma} = \mathrm{col}(\sigma_2, \cdots, \sigma_m)$, $\bar{x} = \mathrm{col}(x_{m+1}, \cdots, x_n)$, and $x = \mathrm{col}(\sigma_1, \bar{\sigma}, \bar{x})$ allows us to represent W, U, and G as

$$W = \begin{pmatrix} w_{11} & W_{21}^T \\ W_{21} & W_{22} \end{pmatrix},\ U = \begin{pmatrix} u_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \end{pmatrix},\ G = \begin{pmatrix} g_{11} & G_{21}^T & G_{31}^T \\ G_{21} & G_{22} & G_{32}^T \\ G_{31} & G_{32} & G_{33} \end{pmatrix},$$

where w_{11}, u_{11} and g_{11} are scalars, $W_{21}, U_{21} \in R^{(m-1)\times 1}$, $U_{12} \in R^{1\times(m-1)}$, $W_{22}, U_{22}, G_{22} \in R^{(m-1)\times(m-1)}$, $U_{31}, G_{31} \in R^{(n-m)\times 1}$, $U_{31}, G_{32} \in R^{(n-m)\times(m-1)}$, and $G_{33} \in R^{(n-m)\times(n-m)}$.

We further denote

$$\bar{K} = \ \mathrm{diag}(\mu_2, \cdots, \mu_m),$$

$$I(\mu_1, \cdots, \mu_m) = \begin{pmatrix} -w_{11}\mu_1^2 - 2u_{11}\mu_1 + g_{11} & -\mu_1(U_{21}^T + W_{21}^T\bar{K}) + G_{21}^T - U_{12}\bar{K} & -\mu_1 U_{31}^T + G_{31}^T \\ -\mu_1(U_{21} + \bar{K}W_{21}) + G_{21} - \bar{K}U_{12}^T & G_{22} - \bar{K}W_{22}\bar{K} - \bar{K}U_{22}^T - U_{22}\bar{K} & G_{32}^T - \bar{K}U_{32}^T \\ -\mu_1 U_{31} + G_{31} & G_{32} - U_{32}\bar{K} & G_{33} \end{pmatrix}.$$

Preparatory to proving Lemma 1, we give the following lemmas.

Lemma 5 *A necessary and sufficient condition for function $V(x)$ of (8) to satisfy condition (9) is that*

$$\det I(\mu_1, \cdots, \mu_m) > 0, \text{ for any } \mu_j \in [0, k_j] \ (j = 1, \cdots, m). \tag{33}$$

Proof. *Necessity:* For any $\mu_j \in [0, k_j]$ $(j = 1, \cdots, m)$, if we set $f_j(\sigma_j) = \mu_j \sigma_j \in K_j[0, k_j]$, then it follows that $\dot{V}_{\mathcal{S}_0} < 0$, that is,

$$\begin{aligned} -\dot{V}_{\mathcal{S}_0} &= \begin{pmatrix} \mu_1\sigma_1 \\ \bar{K}\bar{\sigma} \\ \sigma_1 \\ \bar{\sigma} \\ \bar{x} \end{pmatrix}^T \begin{pmatrix} -w_{11} & -W_{21}^T & -u_{11} & -U_{21}^T & -U_{31}^T \\ -W_{21} & -W_{22} & -U_{12}^T & -U_{22}^T & -U_{32}^T \\ -u_{11} & -U_{12} & g_{11} & G_{21}^T & G_{31}^T \\ -U_{21} & -U_{22} & G_{21} & G_{22} & G_{32}^T \\ -U_{31} & -U_{32} & G_{31} & G_{32} & G_{33} \end{pmatrix} \begin{pmatrix} \mu_1\sigma_1 \\ \bar{K}\bar{\sigma} \\ \sigma_1 \\ \bar{\sigma} \\ \bar{x} \end{pmatrix} \\ &= x^T I(\mu_1, \cdots, \mu_m) x > 0, \quad \forall \|x\| \neq 0. \end{aligned}$$

Hence,

$$I(\mu_1, \cdots, \mu_m) > 0.$$

Consequently,

$$\det I(\mu_1, \cdots, \mu_m) > 0.$$

Sufficiency: Assume that

$$\det I(\mu_1, \cdots, \mu_m) > 0 \text{ for any } \mu_j \in [0, k_j] \ (j = 1, \cdots, m).$$

Since $I(0, \cdots, 0) = G > 0$,

$$I(\mu_1, 0, \cdots, 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = I(0, 0, \cdots, 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0$$

holds[1]. $\det I(\mu_1, 0, \cdots, 0) > 0$ gives

$$I(\mu_1, 0, \cdots, 0) > 0 \text{ for any } \mu_1 \in [0, k_1].$$

From

$$I(\mu_1, \mu_2, 0, \cdots, 0) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = I(\mu_1, 0, 0, \cdots, 0) \begin{pmatrix} 2 \\ 2 \end{pmatrix} > 0$$

and

$$\det I(\mu_1, \mu_2, 0, \cdots, 0) > 0,$$

[1] $T \begin{pmatrix} i \\ j \end{pmatrix}$ denotes a matrix obtained by deleting the i-th row and the j-th column of the matrix T.

we obtain $I(\mu_1, \mu_2, 0, \cdots, 0) > 0$ for any $\mu_j \in [0, k_j]$ $(j = 1, 2)$. Analogously, it follows that

$$I(\mu_1, \mu_2, \cdots, \mu_m) > 0, \text{ for any } \mu_j \in [0, k_j] \ (j = 1, 2, \cdots, m). \tag{34}$$

Letting

$$\mu_j(\sigma_j) = \begin{cases} f_j(\sigma_j)/\sigma_j, & \sigma_j \neq 0, \\ 0, & \sigma_j = 0, \end{cases}$$

for any $f_j(\sigma_j) \in K_j[0, k_j]$ $(j = 1, 2, \cdots, m)$, we have

$$f_j(\sigma_j) = \mu_j(\sigma_j)\sigma_j, \text{ and } \mu_j(\sigma_j) \in [0, k_j] \ (j = 1, 2, \cdots, m).$$

Therefore,

$$-\dot{V}_{\mathcal{S}_0} = x^T I(\mu_1(\sigma_1), \cdots, \mu_m(\sigma_m))x > 0, \ \forall \|x\| \neq 0.$$

That is, $\dot{V}_{\mathcal{S}_0} < 0$. This completes the proof of Lemma 5. ∎

We obtain Lemma 6 in a similar way.

Lemma 6 *If conditions (a) and (b) in Lemma 1 hold, then*

(a) $I(\alpha_1, \mu_2, \cdots, \mu_m) > 0$ $(\alpha_1 = 0, k_1)$ *for any* $\mu_j \in [0, k_j]$ $(j = 2, \cdots, m)$,

(b) $I(\mu_1, 0, \cdots, 0) > 0$ *for any* $\mu_1 \in [0, k_1]$.

Lemma 7 *Let S be an $r \times r$ nonsingular symmetric matrix; let β and δ be r-dimensional vectors; and let μ, a, b and c be real numbers. If we denote*

$$Q(\mu) = \begin{pmatrix} a\mu^2 + 2b\mu + c & \mu\delta^T + \beta^T \\ \mu\delta + \beta & S \end{pmatrix}, \ R = \begin{pmatrix} a & \begin{bmatrix} b & \delta^T \end{bmatrix} \\ \begin{bmatrix} b \\ \delta \end{bmatrix} & Q(0) \end{pmatrix},$$

then we have

$$\det Q(\mu) = \mu^2 \det R \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 2\mu \det R \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \det R \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\Delta = \left[\det R \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]^2 - \left[\det R \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right]\left[\det R \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = -(\det R)(\det S)$$

Proof.

$$\begin{aligned}
\det Q(\mu) &= (\det S)[a\mu^2 + 2b\mu + c - (\mu\delta^T + \beta^T)S^{-1}(\mu\delta + \beta)] \\
&= \mu^2(\det S)(a - \delta^T S^{-1}\delta) + 2\mu(\det S)(b - \delta^T S^{-1}\beta) + (\det S)(c - \beta^T S^{-1}\beta) \\
&= \mu^2 \ \det R \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 2\mu \det R \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \ \det R \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\Delta &= (\det S)^2[(b - \delta^T S^{-1}\beta)]^2 - (a - \delta^T S^{-1}\delta)(c - \beta^T S^{-1}\beta)] \\
&= -(\det S)\left[(\det S) \det \begin{pmatrix} a - \delta^T S^{-1}\delta & b - \delta^T S^{-1}\beta \\ b - \delta^T S^{-1}\beta & c - \beta^T S^{-1}\beta \end{pmatrix}\right] = -(\det R)(\det S).
\end{aligned}$$

Lemma 8 *If conditions (a) and (b) in Lemma 1 hold, then*

$$\det R(\bar{K}) \leq 0 \textit{ for any } \mu_j \in [0, k_j] \ (j = 2, \cdots, m),$$

where

$$R(\bar{K}) = \begin{pmatrix} -w_{11} & \begin{bmatrix} -u_{11} & -U_{21}^T - W_{21}^T \bar{K} & -U_{31}^T \end{bmatrix} \\ \begin{bmatrix} -u_{11} \\ -U_{21} - \bar{K} W_{21} \\ -U_{31} \end{bmatrix} & I(0, \mu_2, \cdots, \mu_m) \end{pmatrix}.$$

Proof. If $\lambda_1 = 0$, then $w_{11} = \lambda_1 c_1^T b_1 = 0$. From $I(0, \mu_2, \cdots, \mu_m) > 0$, we obtain $\det R(\bar{K}) \leq 0$.

If $\lambda_1 > 0$, it is easy to show that $\det R(\bar{K}) \leq 0$. In addition, if $\det R(\bar{K}) > 0$, then $R(\bar{K}) > 0$.

Now we construct a linear system of constant coefficients with $n + 1$ variables as follows:

$$\check{\mathcal{S}}_0 : \begin{cases} \dfrac{dx}{dt} = Ax + \displaystyle\sum_{j=2}^{m} \mu_j b_j x_j + b_1 \xi_1, \\ \dfrac{d\xi_1}{dt} = c_1^T \left(Ax + \displaystyle\sum_{j=2}^{m} \mu_j b_j x_j + b_1 \xi_1 \right), \end{cases} \tag{35}$$

and set

$$V_1(\xi_1, x) = x^T P x + \frac{1}{2} \sum_{j=2}^{m} \mu_j \lambda_j x_j^2 + \frac{1}{2} \lambda_1 \xi_1^2.$$

Then, we obtain

$$-\dot{V}_1|_{\check{\mathcal{S}}_0} = \begin{pmatrix} \xi_1 \\ x \end{pmatrix}^T R(\bar{K}) \begin{pmatrix} \xi_1 \\ x \end{pmatrix} > 0, \quad \forall \ \|x\| + |\xi_1| \neq 0,$$

which contradicts the nonasymptotic stability of the zero solution of system (35). This completes the proof. ∎

Now, we are ready prove Lemma 1.

Proof of Lemma 1:

Necessity: The necessity is clear and thus omitted.

Sufficiency: First, we show that (33) holds. If it does not, there exists $\mu_{j0} \in [0, k_j]$ $(j = 1, 2, \cdots, m)$ such that $\det I(\mu_{10}, \cdots, \mu_{m0}) \leq 0$. Then, from Lemma 6, it follows that $\det I(0, \mu_{20}, \cdots, \mu_{m0}) > 0$. And so there exists $\bar{\mu}_1 \in [0, k_1]$ such that

$$\det I(\bar{\mu}_1, \mu_{20}, \cdots, \mu_{m0}) = 0. \tag{36}$$

Let $\bar{K}_0 = \operatorname{diag}(\mu_{20}, \cdots, \mu_{m0})$, and denote

$$a(\varepsilon) = \det R(\varepsilon \bar{K}_0) \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \; b(\varepsilon) = \det R(\varepsilon \bar{K}_0) \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; c(\varepsilon) = \det R(\varepsilon \bar{K}_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, $a(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$ are polynomials in ε. From Lemmas 7 and 8, we have

$$\det I(\bar{\mu}_1, \varepsilon\mu_{20}, \cdots, \varepsilon\mu_{m0}) = a(\varepsilon)\mu_1^2 + 2b(\varepsilon)\mu_1 + c(\varepsilon)$$

and

$$\Delta(\varepsilon) = b^2(\varepsilon) - a(\varepsilon)c(\varepsilon) = -[\det R(\varepsilon \bar{K}_0)][\det I(0, \varepsilon\mu_{20}, \cdots, \varepsilon\mu_{m0})] \geq 0, \; \forall \varepsilon \in [0, 1].$$

Since $a(1) \neq 0$ (If not, from (36) it follows that

$$\det I(0, \varepsilon\mu_{20}, \cdots, \varepsilon\mu_{m0}) \geq 0 \;\text{ or }\; \det I(\mu_1, \varepsilon\mu_{20}, \cdots, \varepsilon\mu_{m0}) \geq 0,$$

which contradicts Lemma 6, then from (36), we have

$$\bar{\mu}_1 = \frac{-b(1) + \sqrt{b^2(1) - a(1)c(1)}}{a(1)} \text{ or } \bar{\mu}_1 = \frac{-b(1) - \sqrt{b^2(1) - a(1)c(1)}}{a(1)}.$$

Without loss of generality, we suppose that the above holds. Let

$$P(\varepsilon) = \frac{-b(\varepsilon) + \sqrt{b^2(\varepsilon) - a(\varepsilon)c(\varepsilon)}}{a(\varepsilon)}. \tag{37}$$

Then, $P(1) = \bar{\mu}_1$, and

$$\det I(P(\varepsilon), \varepsilon\mu_{20}, \cdots, \varepsilon\mu_{m0}) = a(\varepsilon)P^2(\varepsilon) + 2b(\varepsilon)P(\varepsilon) + c(\varepsilon) = 0.$$

From Lemma 6 ,

$$P(\varepsilon) \neq 0 \text{ and } P(\varepsilon) \neq k_1, \text{ for } \varepsilon \in [0,1].$$

Since $a(\varepsilon)$ is a non-zero polynomial in ε, it has finite zero points only in $[0,1]$. So, we can assume that its zero points are

$$0 \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{s-1} < \varepsilon_s < 1.$$

Lemma 8 says that $P(\varepsilon)$ is continuous on $(\varepsilon_s, 1]$. And since $P(\varepsilon) \neq 0$, $P(\varepsilon) \neq k_1$, and $0 < P(1) = \bar{\mu}_1 < k_1$, we have

$$0 < P(\varepsilon) < k_1, \quad \varepsilon \in (\varepsilon_s, 1].$$

Therefore, $\lim\limits_{\varepsilon \to \varepsilon_s + 0} P(\varepsilon) = P_s$, and $0 < P_s < k_1$.

From (37), it follows that $\lim\limits_{\varepsilon \to \varepsilon_s} P(\varepsilon) = P_s$.

Define $P(\varepsilon_s) = P_s$. Then, $P(\varepsilon)$ is continuous on $\varepsilon \in (\varepsilon_{s-1}, \varepsilon_s]$. In the same way, we can prove that

$$\lim_{\varepsilon \to \varepsilon_{s-1}} P(\varepsilon) = P_{s-1} \text{ and } 0 < P_{s-1} < k_1.$$

Analogously, it follows that

$$P(0) \in [0, k_1] \text{ for } a(0) \neq 0 \text{ or } \lim_{\varepsilon \to 0+} P(\varepsilon) = P_0 \in [0, k_1] \text{ for } a(0) = 0.$$

Assume that

$$\mu_1 = \begin{cases} P(0), & a(0) \neq 0, \\ P_0, & a(0) = 0, \end{cases}$$

Then, $\mu_1 \in [0, k_1]$ and $\det I(\mu_1, 0, \cdots, 0)$, which contradicts Lemma 6. This completes the proof. ∎