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Delay-Dependent Stabilization of Linear Systems with Time-Varying State and Input Delays ⋆

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Abstract

The Integral-Inequality Method is a new way of tackling the delay-dependent stabilization problem for a linear system with time-varying state and input delays: $\dot{x}(t) = Ax(t) + A_1x(t-h_1(t)) + B_1u(t) + B_2u(t-h_2(t))$. In this paper, a new integral inequality for quadratic terms is first established. Then, it is used to obtain a new state- and input-delay-dependent criterion that ensures the stability of the closed-loop system with a memoryless state feedback controller. Finally, some numerical examples are presented to demonstrate that control systems designed based on the criterion are effective, even though neither (A, B_1) nor $(A + A_1, B_1)$ is stabilizable.

Key words: input delays; state delays; delay-dependent stability; robust stabilization; integral inequality; linear matrix inequality (LMI).

1 Introduction

Time delays are frequently encountered in a variety of dynamic systems, such as nuclear reactors, chemical engineering systems, biological systems, and population dynamics models (Kolmanovskii & Nosov (1986); Kuang (1993)). They are often a source of instability and degradation in control performance in many control systems. The analysis of the stability of dynamic control systems with delays and the synthesis of controllers for them are important both in theory and in practice (see Niculescu (2001); Gu *et al.* (2003)), and are thus of interest to a great number of researchers (see Barmish (1985); Xie (1996); Gu (2000); Han (2002) et al). Recently, Richard (2003) summarized current research on time-delay systems and listed four open problems, one of which is the following.

Open Problem 1 *Consider a linear system with both state and input delays:*

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + B_1u(t) + B_2u(t-h). \quad (1)$$

If the pairs (A, B_1) and/or $(A + A_1, B_1)$ are not controllable, how can the term $B_2u(t-h)$ be used to achieve efficient control?

When $A_1 = 0$, the system has an input delay. An easy way to deal with it is to reduce it to an ordinary delay-free system by the Artstein model reduction method (Kwon & Pearson (1980); Artstein (1982); Choi & Chung (1995)). However, the complete transformation can only be obtained for a fully known system. That is, this method is not valid when the system contains a time-varying delay or uncertainties. Furthermore, stabilizing controllers obtained by this method are distributed, and therefore difficult to implement.

For the case $A_1 \neq 0$, Fiagbedzi & Pearson (1986, 1987) designed a feedback controller to stabilize the system (1) by transforming it into an ordinary delay-free system and using the concept of spectral stabilizability. The fact that this method requires that the unstable poles of the system be known exactly makes it difficult to use on a system with a time-varying delay or uncertainties, and

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the resulting controller is also distributed. Choi & Chung (1995), Kim *et al.* (1996) and Han & Mehdi (1998) proposed another method to directly design a robust stabilizing controller for an uncertain system with state and input delays. Their approach involves the design of a memoryless controller to guarantee the stability of the closed-loop system. Since this controller is independent of the delay, it tends to be unduly conservative, especially when the actual delay is small. To the best of our knowledge, surprisingly few delay-dependent conditions have so far been established for the open problem stated above.

This paper proposes a new method called the *Integral-Inequality Method* that can be used to study the delay-dependent stabilization issue of the open problem for time-varying delays. Incorporating Moon *et al.*'s inequality (Moon *et al.* (2001)) and the Leibniz-Newton formula yields an integral inequality for quadratic terms. This is used to obtain a new state- and input-delay-dependent stabilization condition by means of the Lyapunov-Krasovskii functional approach. It is easy to show that the new criterion does not require any assumptions about the system matrices, e.g., neither (A, B_1) nor $(A + A_1, B_1)$ needs to be stabilizable. So, a control system designed based on this criterion is effective, even if neither (A, B_1) nor $(A + A_1, B_1)$ is stabilizable. Moreover, a numerical example shows that applying the new criterion to the system (1) with $B_2 = 0$ yields a less conservative result than those obtained by Fridman & Shaked (2002, 2003) and Gao & Wang (2003).

Notation: Throughout this paper, the superscripts '-1' and 'T' stand for the inverse and transpose of a matrix, respectively; $\Re^n$ denotes an n-dimensional Euclidean space; $\Re^{n\times m}$ is the set of all $n \times m$ real matrices; $P > 0$ means that the matrix P is positive definite; I is an appropriately dimensioned identity matrix; diag$\{\cdots\}$ denotes a block-diagonal matrix; and the symmetric terms in a symmetric matrix are denoted by $*$, e.g., $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$.

2 Problem statement

Consider the following system with time-varying state and input delays:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t - h_1(t)) \\ &\quad + B_1 u(t) + B_2 u(t - h_2(t)),\ t > 0, \\ x(t) &= \phi(t),\ t \in [-\max\{\bar{h}_1, \bar{h}_2\}, 0], \end{aligned} \tag{2}$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are the state and control input, respectively; ϕ is a continuously differential initial function; A, A_1, B_1 and B_2 are known constant real matrices with appropriate dimensions; and $h_1(t)$ and $h_2(t)$ are time-varying bounded delays satisfying

$$0 \le h_1(t) \le \bar{h}_1,\ 0 \le h_2(t) \le \bar{h}_2,\ \dot{h}_1(t) \le d < 1. \tag{3}$$

The memoryless state feedback controller

$$u(t) = Kx(t) \tag{4}$$

is employed to stabilize (2). The objective of this study is to develop a new delay-dependent stabilization method that provides a controller gain, K, as well as upper bounds, $\bar{h}_1$ and $\bar{h}_2$, on the delays such that the resulting closed-loop system, (2) and (4), is asymptotically stable for any $h_1(t)$ and $h_2(t)$ satisfying (3). For this purpose, the following lemmas are first introduced.

Lemma 1 *(Moon et al. (2001)) The following inequality holds for any $a \in \Re^{n_a}$, $b \in \Re^{n_b}$, $N \in \Re^{n_a \times n_b}$, $X \in \Re^{n_a \times n_a}$, $Y \in \Re^{n_a \times n_b}$, and $Z \in \Re^{n_b \times n_b}$:*

$$-2a^T N b \le \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - N \\ * & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{5}$$

where $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \ge 0.$

Applying the above lemma yields the following integral inequality for quadratic terms.

Lemma 2 *Let $x(t) \in \Re^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices X, M_1, $M_2 \in \Re^{n\times n}$ and $Z \in \Re^{2n\times 2n}$, and a scalar function $h := h(t) \ge 0$:*

$$-\int_{t-h}^{t} \dot{x}^T(s) X \dot{x}(s) ds \le \xi^T(t) \Upsilon \xi(t) + h \xi^T(t) Z \xi(t), \tag{6}$$

where

$$\Upsilon := \begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_2 \end{bmatrix},\quad \xi(t) := \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

$$\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \ge 0, \tag{7}$$

with $Y := [M_1\ M_2]$.

PROOF. From the Leibniz-Newton formula,

$$0 = x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(s)ds. \tag{8}$$

So, the following equation holds for any $N_1, N_2 \in \Re^{n\times n}$.

$$\begin{aligned} 0 &= 2\left[x^T(t)N_1^T + x^T(t-h)N_2^T\right] \\ &\quad \times \left[x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(s)ds\right] \\ &= 2\xi^T(t)N^T\left[I \quad -I\right]\xi(t) - 2\int_{t-h}^{t} \xi^T(t)N^T\dot{x}(s)ds. \end{aligned} \tag{9}$$

where $N := [N_1 \quad N_2]$. Applying Lemma 1 with $a := \dot{x}(s)$ and $b := \xi(t)$ yields

$$\begin{aligned} -2\int_{t-h}^{t} \xi^T(t)N^T\dot{x}(s)ds &\le \int_{t-h}^{t} \dot{x}^T(s)X\dot{x}(s)ds \\ &\quad +2\xi^T(t)(Y^T - N^T)[I \quad -I]\xi(t) + h\xi^T(t)Z\xi(t). \end{aligned} \tag{10}$$

Substituting (10) into (9) gives us

$$\begin{aligned} &-\int_{t-h}^{t} \dot{x}^T(s)X\dot{x}(s)ds \\ &\le 2\xi^T(t)Y^T\left[I \quad -I\right]\xi(t) + h\xi^T(t)Z\xi(t) \end{aligned} \tag{11}$$

After a simple rearrangement, (11) yields (6). This completes the proof. $\square$

Remark 1 *(6) is called an integral inequality. It plays a key role in the derivation of a criterion for delay-dependent stabilization in this paper. Note that the free matrices N_1 and N_2 introduced in the proof do not appear in the integral inequality. The conservatism of the descriptor model transformation method, which is closely related to free parameters, is discussed in the next section.*

Remark 2 *The integral inequality (6) is quite different from the ones used in Gu (2000). The free terms in (6), for example, M_1 and M_2, help in the design of the controller (4) for the system (2), but the integral inequalities in Gu (2000) do not.*

The integral inequality (6) holds under the inequality constraint (7). When $X > 0$, the constraint condition can be removed from the integral inequality. For example, taking $Z = Y^TX^{-1}Y$ guarantees (7) because

$$\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ * & Y^TX^{-1}Y \end{bmatrix} = G^TG \ge 0,$$

where $G := \begin{bmatrix} X^{1/2} & X^{-1/2}Y \\ 0 & 0 \end{bmatrix}$. This yields the following proposition.

Proposition 3 *Let $x(t) \in \Re^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $M_1, M_2 \in \Re^{n\times n}$ and $X=X^T > 0$, and a scalar function $h := h(t) \ge 0$:*

$$\begin{aligned} &-\int_{t-h}^{t} \dot{x}^T(s)X\dot{x}(t)ds \le \\ &\qquad \xi^T(t)\begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_2 \end{bmatrix}\xi(t) \\ &\qquad +h\xi^T(t)\begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix}X^{-1}\begin{bmatrix} M_1 & M_2 \end{bmatrix}\xi(t), \end{aligned} \tag{12}$$

where $\xi(t)$ is defined in Lemma 2. $\square$

3 Main results

This section presents the delay-dependent stabilization conditions obtained by means of the integral-inequality method.

The closed-loop system constructed by means of (2) and (4) is given by

$$\dot{x}(t) = A_Kx(t) + A_1x(t-h_1(t)) + B_Kx(t-h_2(t)), \tag{13}$$

where

$$A_K = A + B_1K, \ B_K = B_2K. \tag{14}$$

The following theorem is obtained for the system (13).

Theorem 4 *For given numbers $\lambda_i, \mu_i, i = 1, 2$, if there exist positive matrices $\bar{P} > 0$, $\bar{R}_1 > 0$, $\bar{R}_2 > 0$, and $\bar{Q} > 0$ such that the following LMI holds:*

$$\Psi := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & 0 & 0 & \bar{P} \\ * & \Sigma_{22} & 0 & \Sigma_{24} & \Sigma_{25} & \bar{h}_1\bar{R}_1 & 0 & 0 \\ * & * & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & 0 & \bar{h}_2\bar{R}_2 & 0 \\ * & * & * & -\bar{h}_1\bar{R}_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{h}_2\bar{R}_2 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{h}_1\bar{R}_1 & 0 & 0 \\ * & * & * & * & * & * & -\bar{h}_2\bar{R}_2 & 0 \\ * & * & * & * & * & * & * & -\bar{Q} \end{bmatrix} < 0 \tag{15}$$

where

$$
\begin{aligned}
\Sigma_{11} &= A\bar{P} + \bar{P}A^T + B_1Y + Y^TB_1^T \\
&\quad -\lambda_1\mu_1^{-1}(A_1\bar{Q} + \bar{Q}A_1^T) - \lambda_2\mu_2^{-1}(B_2Y + Y^TB_2^T) \\
&\quad -\lambda_1^2\mu_1^{-2}(1-d)\bar{Q}, \\
\Sigma_{12} &= \mu_1^{-1}A_1\bar{Q} + \bar{P} + \lambda_1\mu_1^{-1}\bar{Q} + \lambda_1\mu_1^{-2}(1-d)\bar{Q}, \\
\Sigma_{13} &= \mu_2^{-1}B_2Y + \bar{P} + \lambda_2\mu_2^{-1}\bar{P}, \\
\Sigma_{14} &= \bar{h}_1(Y^TB_1^T + \bar{P}A^T - \lambda_1\mu_1^{-1}\bar{Q}A_1^T - \lambda_2\mu_2^{-1}Y^TB_2^T), \\
\Sigma_{15} &= \bar{h}_2(Y^TB_1^T + \bar{P}A^T - \lambda_1\mu_1^{-1}\bar{Q}A_1^T - \lambda_2\mu_2^{-1}Y^TB_2^T), \\
\Sigma_{22} &= -2\mu_1^{-1}\bar{Q} - \mu_1^{-2}(1-d)\bar{Q}, \\
\Sigma_{24} &= \bar{h}_1\mu_1^{-1}\bar{Q}A_1^T, \\
\Sigma_{25} &= \bar{h}_2\mu_1^{-1}\bar{Q}A_1^T, \\
\Sigma_{33} &= -2\mu_2^{-1}\bar{P}, \\
\Sigma_{34} &= \bar{h}_1\mu_2^{-1}Y^TB_2^T, \\
\Sigma_{35} &= \bar{h}_2\mu_2^{-1}Y^TB_2^T,
\end{aligned}
$$

then the closed-loop system (13) is asymptotically stable and the state feedback control law is given by

$$u(t) = Y\bar{P}^{-1}x(t). \tag{16}$$

PROOF. Choose a Lyapunov-Krasovskii functional candidate as follows:

$$
\begin{aligned}
V(t) &= x^T(t)Px(t) + \sum_{j=1}^{2} \int_{-\bar{h}_j}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R_j\dot{x}(s)dsd\theta \\
&\quad + \int_{t-h_1(t)}^{t} x^T(s)Qx(s)ds,
\end{aligned}
$$

where $P > 0$, $Q > 0$, $R_1 > 0$, and $R_2 > 0$. Then, the time derivative of $V(t)$ along the trajectory (13) satisfies

$$
\begin{aligned}
\dot{V}(t) &= 2x^T(t)P\dot{x}(t) + \sum_{j=1}^{2} \bar{h}_j\dot{x}^T(t)R_j\dot{x}(t) \\
&\quad -(1-\dot{h}_1(t))x^T(t-h_1(t))Qx(t-h_1(t)) \\
&\quad +x^T(t)Qx(t) - \sum_{j=1}^{2}\int_{t-\bar{h}_j}^{t} \dot{x}^T(s)R_j\dot{x}(s)ds.
\end{aligned}
\tag{17}
$$

From (3), it is clear that the following is true for $j = 1, 2$:

$$-\int_{t-\bar{h}_j}^{t} \dot{x}^T(s)R_j\dot{x}(s)ds \le -\int_{t-h_j(t)}^{t} \dot{x}^T(s)R_j\dot{x}(s)ds. \tag{18}$$

Applying the integral inequality (12) to the term on the right side of (18) for any M_{1j}, $M_{2j} \in \Re^{n\times n}$ yields the following integral inequality for $j = 1, 2$:

$$
\begin{aligned}
-\int_{t-h_j(t)}^{t} \dot{x}^T(s)R_j\dot{x}(t)ds &\le \\
&\eta_j^T(t)\begin{bmatrix} M_{1j}^T + M_{1j} & -M_{1j}^T + M_{2j} \\ * & -M_{2j}^T - M_{2j}\end{bmatrix}\eta_j(t) \\
&+\bar{h}_j\eta_j^T(t)\begin{bmatrix} M_{1j}^T \\ M_{2j}^T\end{bmatrix} R_j^{-1}\begin{bmatrix} M_{1j} & M_{2j}\end{bmatrix}\eta_j(t),
\end{aligned}
\tag{19}
$$

where $\eta_j^T(t) = \begin{bmatrix} x^T(t) & x^T(t-h_j(t))\end{bmatrix}$, $j = 1, 2$. Substituting (18) and (19) into (17), carrying out some algebraic manipulations, and rearranging the terms gives

$$
\begin{aligned}
\dot{V}(t) &\le \xi^T(t)\left[H + \sum_{j=1}^{2}\bar{h}_j\Gamma_1^TR_j\Gamma_1\right]\xi(t) \\
&\quad +\xi^T(t)\left[+\bar{h}_1\Gamma_2^TR_1^{-1}\Gamma_2 + \bar{h}_2\Gamma_3^TR_2^{-1}\Gamma_3\right]\xi(t),
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
\xi^T(t) &= \begin{bmatrix} x^T(t) & x^T(t-h_1(t)) & x^T(t-h_2(t))\end{bmatrix}, \\
H &= \begin{bmatrix} H_{11} & PA_1 - M_{11}^T + M_{21} & PB_K - M_{12}^T + M_{22} \\ * & -(1-d)Q - M_{21}^T - M_{21} & 0 \\ * & * & -M_{22}^T - M_{22}\end{bmatrix}, \\
H_{11} &= PA_K + A_K^TP + Q + M_{11}^T + M_{11} + M_{12}^T + M_{12}, \\
\Gamma_1 &= [\,A_K \;\; A_1 \;\; B_K\,], \\
\Gamma_2 &= [\,M_{11} \;\; M_{21} \;\; 0\,], \\
\Gamma_3 &= [\,M_{12} \;\; 0 \;\; M_{22}\,].
\end{aligned}
\tag{21}
$$

From (20), we find that, if the following matrix inequality holds:

$$
\Xi := \begin{bmatrix}
H & \bar{h}_1\Gamma_1^T & \bar{h}_2\Gamma_1^T & \bar{h}_1\Gamma_2^T & \bar{h}_2\Gamma_3^T \\
* & -\bar{h}_1R_1^{-1} & 0 & 0 & 0 \\
* & * & -\bar{h}_2R_2^{-1} & 0 & 0 \\
* & * & * & -\bar{h}_1R_1 & 0 \\
* & * & * & * & -\bar{h}_2R_2
\end{bmatrix} < 0,
\tag{22}
$$

then applying the Schur complement (Bernussou *et al.* (1989)) yields $\dot{V}(t) < 0$. Thus, by using the Lyapunov-Krasovskii functional theorem (Proposition 5.2 in Gu *et al.* (2003)), we can conclude that (13) is asymptotically stable.

In order to obtain a controller gain, K, from the nonlinear matrix inequality (22) (the nonlinearities come from $R_i^{-1},\ i=1,2$), we first let

$$W=\begin{bmatrix} P & 0 & 0\\ M_{11} & M_{21} & 0\\ M_{12} & 0 & M_{22}\end{bmatrix},\ \bar{A}=\begin{bmatrix} A_K & A_1 & B_K\\ I & -I & 0\\ I & 0 & -I\end{bmatrix}.$$

Then,

$$H=W^T\bar{A}+\bar{A}^TW+\mathrm{diag}\{\,Q,\quad -(1-d)Q,\quad 0\,\},$$
$$\Gamma_2^T=W^T\begin{bmatrix}0\\ I\\ 0\end{bmatrix},\ \Gamma_3^T=W^T\begin{bmatrix}0\\ 0\\ I\end{bmatrix}.$$

Now, consider the case in which $M_{11}=\lambda_1P,\ M_{12}=\lambda_2P,\ M_{21}=\mu_1Q,\ M_{22}=\mu_2P,\ \mu_1\neq 0$, and $\mu_2\neq 0$. In this case, W is invertible; and

$$W^{-1}=\begin{bmatrix} P^{-1} & 0 & 0\\ -\lambda_1\mu_1^{-1}Q^{-1} & \mu_1^{-1}Q^{-1} & 0\\ -\lambda_2\mu_2^{-1}P^{-1} & 0 & \mu_2^{-1}P^{-1}\end{bmatrix}. \qquad (23)$$

Let $T=\mathrm{diag}\{W^{-1},I,I,R_1^{-1},R_2^{-1}\}$. Then,

$$T^T\Xi T=\begin{bmatrix} H_T & \bar{h}_1W^{-T}\Gamma_1^T & \bar{h}_2W^{-1}\Gamma_1^T & \bar{h}_1\Pi_1^T & \bar{h}_2\Pi_2^T\\ * & -\bar{h}_1R_1^{-1} & 0 & 0 & 0\\ * & * & -\bar{h}_2R_2^{-1} & 0 & 0\\ * & * & * & -\bar{h}_1R_1^{-1} & 0\\ * & * & * & * & -\bar{h}_2R_2^{-1}\end{bmatrix} \qquad (24)$$

where

$$\begin{aligned} H_T &= \bar{A}W^{-1}+W^{-T}\bar{A}^T\\ &\quad +W^{-T}\mathrm{diag}\{Q,\ -(1-d)Q,\ 0\}W^{-1},\\ \Pi_1 &= [\,0\ R_1^{-1}\ 0\,],\\ \Pi_2 &= [\,0\ 0\ R_2^{-1}\,].\end{aligned}$$

After substituting (14) and (23) into (24), setting $\bar{P}=P^{-1}$, $\bar{R}_1=R_1^{-1}$, $\bar{R}_2=R_2^{-1}$, $\bar{Q}=Q^{-1}$, and $Y=KP^{-1}$, and performing some simple algebraic manipulations, we find that if LMI (15) holds, the Schur complement ensures that $T^T\Xi T<0$, and thus $\Xi<0$. So, the resulting closed-loop system (13) is asymptotically stable, and the desired controller is defined by (4) with $K=Y\bar{P}^{-1}$. This completes the proof. □

Free matrices are often introduced in the derivation of delay-dependent stabilization criteria for a system with a state delay. Since they are free, they should not be subject to any constraints. However, they cannot ultimately be eliminated from the conditions in existing criteria; and as a result, they are in fact subject to constraints. In contrast, the condition in Theorem 4 contains no free matrices at all. This is the main reason why it produces less conservative results than existing methods. To illustrate this point, we compare Theorem 4 with the descriptor model transformation method in Fridman & Shaked (2002, 2003) and Gao & Wang (2003).

Consider thesystem

$$\dot{x}(t)=Ax(t)+A_1x(t-h)+B_1u(t) \qquad (25)$$

in Fridman & Shaked (2002, 2003) and Gao & Wang (2003), which is a special case of (2) ($B_2=0$). In order to derive a stabilization condition for the state feedback $u(t)=Kx(t)$, the descriptor model transformation method introduces the following zero term into the derivative of the Lyapunov-Krasovskii functional:

$$0=2\left[x^T(t)P_2^T+\dot{x}^T(t)P_3^T\right]\times \left[-\dot{x}(t)+(A+B_1K+A_1)x(t)-A_1\int_{t-h}^{t}\dot{x}(s)ds\right] \qquad (26)$$

where P_2 and P_3 are free matrices. Moon *et al.*'s inequality is applied to bound the cross term and to derive some stabilization conditions. Since P_2 and P_3 are free, they should not appear in the bounding term, and should not be subject to any constraints. However, the free matrices in (19a) of Fridman & Shaked (2002) and (36) of Fridman & Shaked (2003) are restricted to $Q_2+Q_2^T<0$ and $Q_3+Q_3^T>0$, where $Q_2=-P_3^{-1}P_2P_1^{-1}$ and $Q_3=P_3^{-1}$ with $P_1>0$. Similar conditions are also imposed in Gao & Wang (2003). This is the main reason for the conservatism of the descriptor model transformation method. On the other hand, for (25), Eq. (26) is equivalent to

$$\begin{aligned} 0 &= 2\left[x^T(t)P_2^T+\{(A+B_1K)x(t)+A_1x(t-h)\}^TP_3^T\right]\\ &\quad\times\left[A_1x(t)-A_1x(t-h)-A_1\int_{t-h}^{t}\dot{x}(s)ds\right]\\ &= 2\left[x^T(t)N_1^T+x^T(t-h)N_2^T\right]\\ &\quad\times\left[x(t)-x(t-h)-\int_{t-h}^{t}\dot{x}(s)ds\right],\end{aligned}$$

where $N_1^T=[P_2^T+(A+B_1K)^TP_3^T]A_1$ and $N_2^T=A_1^TP_3^TA_1$. When the cross term is bounded using Lemma 2, as stated in Remark 1, neither of the free matrices, P_2 and P_3, in N_1 and N_2 appears in the result. So, in general, the integral-inequality method produces less conservative results than the descriptor model transformation method.

Theorem 4 employs four tuning parameters: λ_i and μ_i $(i = 1, 2)$. One way to adjust them is as follows. First, a consideration of (15) yields $\mu_2 > 0$, $\mu_1 > -(1-d)/2$, and $\mu_1 \neq 0$. From (21), we know that the choice of $\lambda_i < 0$ $(i = 1, 2)$ increases the degree of stability of H_{11} defined in (21). So, the tuning parameters are chosen under the following condition:

$$\begin{aligned} &\lambda_1 < 0,\ \lambda_2 < 0, \\ &-\mu_1 < (1-d)/2,\ \mu_1 \neq 0,\ \text{and}\ -\mu_2 < 0 \end{aligned} \tag{27}$$

and we define $x = [\lambda_1\ \lambda_2\ \mu_1\ \mu_2]^T$. Next, we choose the cost function to be $f(x) = t_{min}$, for which $\Psi \leq t_{min}I$, where Ψ is defined in (15). The scalar parameter t_{min}, which is a function of x, is obtained by solving the feasibility problem with the solver `feasp` in the LMI Toolbox (Version 1.0.8, The MathWorks (1995)). It is positive when there exists no feasible solution to the set of LMIs under consideration. Finally, applying a numerical optimization algorithm, such as `fmincon` in the Optimization Toolbox (Version 3, The MathWorks (2004)), to $f(x)$ under the constraint (27) yields a locally convergent solution to the problem. If the resulting minimum value of the cost function is negative, then the tuning parameters that solve the problem are found. This method is summarized in the following algorithm.

Algorithm (Maximizing $\bar{h}_1 > 0$ for a fixed $\bar{h}_2 > 0$):

- Step 1: Set a step length, h_{step}, for $\bar{h}_1$. Choose an upper bound, ub, and a lower bound, lb, on x satisfying (27). Select the initial values x_0 for x and h_{10} for $\bar{h}_1$ (where h_{10} is sufficiently small). In addition, our experience shows that choosing $x_0 = [-1,\ -1,\ 1,\ 1]^T$ works in a large number of cases. Solve the following problem

$$\min_x f(x),\quad \text{subject to}\ \ (27) \tag{28}$$

  using the function `fmincon` with x_0, h_{10}, ub, and lb; and obtain a new value for the parameter vector x. If $f(x) < 0$, go to Step 2; otherwise, stop.
- Step 2: Let $x_0 = x$ and $h_{10} = h_{10} + h_{step}$; and solve problem (28) again using the function `fmincon` with the new x_0, h_{10}, ub, and lb.
- Step 3: If $f(x) < 0$, go to Step 2; otherwise, stop.

Remark 3 *For the above algorithm, a smaller step length for $\bar{h}_1$ results directly in an $\bar{h}_1$ with a higher accuracy; but the price we pay is an increase in computation time. To keep the computation time down, we can obtain a suitable $\bar{h}_1$ with a higher accuracy in two steps: First, choose a relatively large step length, e.g., $h_{step} = 0.1$, to solve (28) using the above algorithm and obtain an $\bar{h}_1$ with a low accuracy and the corresponding parameters $x = [\lambda_1\ \lambda_2\ \mu_1\ \mu_2]^T$. Then, use these parameters to solve (15), and thus obtain an $\bar{h}_1$ with a higher accuracy.*

Remark 4 *The criterion in Theorem 4 does not require any assumptions about the system matrices, e.g., the pairs (A, B_1) and $(A + A_1, B_1)$ need not be stabilizable. So, systems designed based on this criterion that have both state and input delays can be stabilized, even when neither (A, B_1) nor $(A + A_1, B_1)$ is stabilizable.*

Remark 5 *Theorem 4 employs the integral inequality (12). Employing (6) and (7) to bound (18) yields a more general result, but it makes the condition more complicated. On the other hand, when the symbol "$\geq$" in (7) is replaced by "$>$", it can easily be shown that the result obtained by using (6) and (7) to bound (18) is the same as that obtained by using (12) in combination with the variable elimination technique (Gu (2001)).*

4 Numerical Examples

This section presents numerical examples that demonstrate the validity of the method described above.

Example 5 *Consider the following system:*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t-h_1(t)) \\ &\quad + B_1u(t) + B_2u(t-h_2(t)), \end{aligned} \tag{29}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ -0.5 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} -2 & -0.5 & 0 & 0 \\ -0.2 & -1 & 0 & 0 \\ 0.5 & 0 & -2 & -0.5 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$
$$B_1 = [\,1\ 1\ 1\ 0\,]^T,\qquad B_2 = [\,0\ 1\ 1\ 1\,]^T.$$

and there are two constant delays satisfying $0 \leq h_i \leq \bar{h}_i, i = 1, 2$.

It is clear that neither (A, B_1) nor $(A+A_1, B_1)$ of (29) is stabilizable. In spite of that, applying Theorem 4 yields a memoryless state feedback control law, $u(t) = Kx(t)$, that stabilizes the system (29). The algorithm in Section 3 was used to find a maximum $\bar{h}_1$ for $\bar{h}_2 = 0.1$. Taking the initial values of the parameters to be $x_0 = [\lambda_1\ \lambda_2\ \mu_1\ \mu_2]^T = [-1\ -1\ 1\ 1]^T$ and $h_{10} = 0.2$, setting the step length to $h_{step} = 0.1$ and choosing the upper and lower bounds on x to be $ub = [-0.01\ \ -0.01\ 5\ 5]^T$ and $lb = [-4\ -4\ 0.1\ 0.1]^T$, respectively, yielded a locally optimal combination: $\lambda_1 = -1.8953$, $\lambda_2 = -1.4451$, $\mu_1 = 2.7388$ and $\mu_2 = 1.3654$, which gave the maximum value $\bar{h}_1 = 0.56$. The corresponding control law was $K =$

Table 1
Upper bound, $\bar{h}$, and corresponding state feedback control law, K, for system (30).

Method	$\bar{h}$	K
Fridman & Shaked (2003)	1.408	Not provided
Fridman & Shaked (2002)	1.510	$[-58.31 \quad -294.9]$
Gao & Wang (2003)	3.200	$[-7.964 \quad -14.77]$
Theorem 4	6.000	$[-70.18 \quad -77.67]$

$[0.0129\ -0.0031\ -0.0009\ -0.3181]$. However, no delay-independent state feedback control law can be found by using the methods in Choi & Chung (1995); Kim *et al.* (1996); Han & Mehdi (1998). That is, their methods are inapplicable to this example.

Example 6 *Consider the following system:*

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + Bu(t), \tag{30}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and there is a constant delay, h, satisfying $0 \leq h \leq \bar{h}$.

(30) contains only a state delay. Fridman & Shaked (2002, 2003) and Gao & Wang (2003) calculated the upper bound $\bar{h}$ for which a state feedback control law, K, exists to stabilize (30). Their results are listed in Table 1 along with the results obtained by Theorem 4 for $\lambda_1 = -0.11$ and $\mu_1 = 0.01$. Clearly, our method produces much less conservative results, thus demonstrating its validity.

This example shows that Theorem 4, which employs an integral inequality, produces much less conservative results than the descriptor model transformation method in Fridman & Shaked (2002, 2003) and Gao & Wang (2003).

5 Conclusion

This paper has presented a state- and input-delay-dependent stabilization criterion for a system with both state and input delays that employs a memoryless state feedback control law. The stabilizing control law is obtained by using the Lyapunov-Krasovskii functional approach combined with an integral inequality. The criterion thus obtained does not require any additional assumptions about the system matrices, for example, that the pairs (A, B_1) and $(A + A_1, B_1)$ be stabilizable. So, the designed control law for a system with both state and input delays is effective, even when neither (A, B_1) nor $(A + A_1, B_1)$ is stabilizable. Numerical examples illustrate the design procedure and show that the criterion is less conservative than existing ones. Moreover, the proposed method can easily be applied to a delay-system with uncertainties to yield a delay-dependent robust stabilization condition.

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