Delay-Dependent Criteria for Robust Stability of Time-Varying Delay Systems

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Abstract

This paper deals with the problem of delay-dependent robust stability for systems with time-varying structured uncertainties and time-varying delays. Some new delay-dependent stability criteria are devised by taking the relationship between the terms in the Leibniz-Newton formula into account. Since free weighting matrices are used to express this relationship and since appropriate ones are selected by means of linear matrix inequalities, the new criteria are less conservative than existing ones. Numerical examples suggest that the proposed criteria are effective and are an improvement over previous ones.

Key words: delay-dependent criteria; robust stability; time-varying delay; time-varying structured uncertainties; linear matrix inequality.

1 Introduction

Stability criteria for time-delay systems have been attracting the attention of many researchers. They can be classified into two categories: delay-dependent and delay-independent criteria. Since delay-dependent criteria make use of information on the length of delays, they are less conservative than delay-independent ones. For delay-dependent criteria (see, for example, Su & Huang

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(1992), Li & de Souza (1997), Gu *et al.* (1998), Cao *et al.* (1998), de Souze & Li (1999), Park (1999), Han & Gu (2001), Kim (2001), Moon *et al.* (2001), Han (2002a, 2002b), Yue & Won (2002), Fridman & Shaked (2002, 2003)), the main approaches currently consist of four model transformations of the original system (Fridman and Shaked (2003)). The first type is a first-order transformation. Since additional eigenvalues are introduced into the transformed system, it is not equivalent to the original one (Gu & Niculescu (2000)). The second type is a neutral transformation. The system obtained by this method is not equivalent to the original one, either (Gu & Niculescu (2001)); and this method requires an additional assumption to obtain the stability condition for the system. In addition, the inequality used to determine the stability of the system is $-2a^T b \leq a^T X a + b^T X^{-1} b,\ \ a, b \in R^n,\ \ X > 0$, which is known to be conservative. Park (1999) introduced a free matrix, M, to obtain a less conservative inequality $-2a^T b \leq (a + Mb)^T X (a + Mb) + b^T X^{-1} b + 2b^T M b$, and Moon *et al.* (2001) extended it to a more general form, $-2a^T b \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - I \\ Y^T - I & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix},\ \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0$. The third type of model transformation employs these inequalities and yields a transformed system that is equivalent to the original one. However, in the derivative of the Lyapunov functional, Park (1999) and Moon *et al.* (2001) used the Leibniz-Newton formula and just replaced some of the terms $x(t - \tau)$ with $x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$ in the derivative of the Lyapunov functional in order to make it easy to handle. For example, in Moon *et al.* (2001), $x(t-\tau)$ was replaced with $x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$ in the expression $2x^T(t)PA_1\dot{x}(t)$, but not in $\tau\dot{x}^T(t)Z\dot{x}(t)$. Since both $x(t - \tau)$ and $x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$ affect the result, there must be some relationship between them; and there must exist optimal weighting matrices for those terms. However, they did not give a method for determining them, but just selected some fixed weighting matrices. Fridman & Shaked (2002, 2003) combined a descriptor model transformation (Fridman (2001)) with Park and Moon's inequalities to yield the fourth type of transformation. This method produces less conservative criteria. However, since the basic approach in Fridman and Shaked (2002, 2003) is also based on the substitution of $x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$ for $x(t - \tau)$, it does not entirely overcome the conservatism of the methods given by Park (1999) and Moon *et al.* (2001).

This paper presents new criteria based on a new method with some interesting features. First, it deals with the system model directly and does not employ any system transformation, thus avoiding the conservatism that results from such a transformation. Second, it does not use the above inequality or the improved inequality to estimate the upper bound of $-2a^T b$. This also reduces the conservatism in the derivation of the stability condition. Third, some free weighting matrices are employed to express the influence of the terms in the Leibniz-Newton formula, in contrast to existing methods, which preselect fixed ones. The matrices are determined by solving linear matrix inequalities (LMIs). This is the main advantage of our method, and is the essential difference between existing methods and ours. Compared with Moon *et al.* (2001), and Fridman & Shaked (2002), our new criteria overcome some of the main sources of conservatism, and contain the criteria in Moon *et al.* (2001) as a special case. Furthermore, the new criteria also contain the well-known delay-independent stability condition in Gu *et al.* (2003) and Hale & Verduyn Lunel (1993). For two examples studied numerically, the new criteria are shown to be effective, offering significant improvements over previously published criteria.

2 Preliminaries

Consider a nominal system Σ_0 with a time-varying delay given by

$$\Sigma_0 : \begin{cases} \dot{x}(t) = Ax(t) + Bx(t - d(t)),\ t > 0 \\ x(t) = \phi(t), t \in [-\tau, 0], \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state vector. The time delay, $d(t)$, is a time-varying continuous function that satisfies

$$0 \le d(t) \le \tau,\ \dot{d}(t) \le \mu < 1, \tag{2}$$

where τ and μ are constants and the initial condition, $\phi(t)$, is a continuous vector-valued initial function of $t \in [-\tau, 0]$.

When the system contains time-varying structured uncertainties, it can be described by

$$\Sigma_1 : \begin{cases} \dot{x}(t) = (A + \Delta A(t))x(t) \\ \qquad\quad +(B + \Delta B(t))x(t - d(t)),\ t > 0, \\ x(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{3}$$

The uncertainties are assumed to be of the form

$$[\Delta A(t)\ \Delta B(t)] = DF(t)[E_a\ E_b], \tag{4}$$

where D, E_a and E_b are constant matrices with appropriate dimensions, and $F(t)$ is an unknown, real, and possibly time-varying matrix with Lebesgue-measurable elements satisfying

$$F^T(t)F(t) \leq 1,\ \forall t. \tag{5}$$

The following lemma is employed to handle the time-varying structured uncertainties in the system.

Lemma 1 *(Xie (1996)) Given matrices $Q = Q^T, H, E$ and $R = R^T > 0$ of appropriate dimensions,*

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying $F^T F \leq R$, if and only if there exists some $\lambda > 0$ such that

$$Q + \lambda H H^T + \lambda^{-1} E^T R E < 0.$$

The Lyapunov functional candidates for Σ_0 and Σ_1 are chosen to have the same form and are given by

$$\begin{aligned} V(x_t) := {} & x^T(t)Px(t) + \int_{t-d(t)}^{t} x^T(s)Qx(s)ds \\ & + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta, \end{aligned} \tag{6}$$

where $P = P^T > 0,\ Q = Q^T > 0$ and $Z = Z^T > 0$ are to be determined.

3 Main results

First, the nominal system, Σ_0, is discussed. The Leibniz-Newton formula is employed to obtain a delay-dependent condition, and the relationship between the terms in the formula is taken into account. Specifically, the terms on the left side of the equation

$$2\left[x^T(t)Y + x^T(t-d(t))T\right] * \left[x(t) - \int_{t-d(t)}^{t} \dot{x}(s)ds - x(t-d(t))\right] = 0 \tag{7}$$

are added to the derivative of the Lyapunov functional, $\dot{V}(x_t)$. In this equation, the free weighting matrices Y and T indicate the relationship between the terms in the Leibniz-Newton formula. As is shown in the following theorem, they can easily be determined by solving the corresponding linear matrix inequalities.

Theorem 2 *Given scalars $\tau > 0$ and $\mu < 1$, the nominal system Σ_0 is asymptotically stable if there exist symmetric positive definite matrices $P = P^T > 0$, $Q = Q^T > 0$ and $Z = Z^T > 0$, a symmetric semi-positive definite matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$, and any appropriately dimensioned matrices Y and T such that the following LMIs are true.*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \tau A^T Z \\ \Phi_{12}^T & \Phi_{22} & \tau B^T Z \\ \tau Z A & \tau Z B & -\tau Z \end{bmatrix} < 0, \tag{8}$$

$$\Psi = \begin{bmatrix} X_{11} & X_{12} & Y \\ X_{12}^T & X_{22} & T \\ Y^T & T^T & Z \end{bmatrix} \geq 0, \tag{9}$$

where

$$\Phi_{11} = PA + A^T P + Y + Y^T + Q + \tau X_{11},$$
$$\Phi_{12} = PB - Y + T^T + \tau X_{12},$$
$$\Phi_{22} = -T - T^T - (1-\mu)Q + \tau X_{22}.$$

PROOF. Using the Leibniz-Newton formula, we can write

$$x(t-d(t)) = x(t) - \int_{t-d(t)}^{t} \dot{x}(s)ds. \tag{10}$$

Then, for any appropriately dimensioned matrices Y and T, we have Eq. (7). On the other hand, for any semi-positive definite matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$, the following holds.

$$\tau\xi^T(t)X\xi(t) - \int_{t-d(t)}^{t} \xi^T(t)X\xi(t)ds \geq 0, \tag{11}$$

where $\xi(t) = [x^T(t) \;\; x^T(t-d(t))]^T$. Then, for $X = X^T \geq 0$, and any matrices Y and T, using Eqs. (7) and (11) and calculating the derivative of $V(x_t)$ in (6) for Σ_0 yields

$$\begin{aligned}
\dot{V}(x_t) &= x^T(t)[PA + A^TP]x(t) \\
&\quad +2x^T(t)PBx(t-d(t)) + x^T(t)Qx(t) \\
&\quad -(1-\dot{d}(t))x^T(t-d(t))Qx(t-d(t)) \\
&\quad +\tau\dot{x}^T(t)Z\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}^T(s)Z\dot{x}(s)ds \\
&\leq x^T(t)[PA + A^TP]x(t) \\
&\quad +2x^T(t)PBx(t-d(t)) + x^T(t)Qx(t) \\
&\quad -(1-\mu)x^T(t-d(t))Qx(t-d(t)) \\
&\quad +\tau\dot{x}^T(t)Z\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s)Z\dot{x}(s)ds \\
&\quad +2\left[x^T(t)Y + x^T(t-d(t))T\right] * \\
&\quad \left[x(t) - \int_{t-d(t)}^{t} \dot{x}(s)ds - x(t-d(t))\right] \\
&\quad +\tau\xi^T(t)X\xi(t) - \int_{t-d(t)}^{t} \xi^T(t)X\xi(t)ds \\
&:= \xi^T(t)\Xi\xi(t) - \int_{t-d(t)}^{t} \zeta^T(t,s)\Psi\zeta(t,s)ds,
\end{aligned} \tag{12}$$

where

$$\zeta(t,s) := [x^T(t) \;\; x^T(t-d(t)) \;\; \dot{x}^T(s)]^T,$$

$$\Xi := \begin{bmatrix} \Phi_{11} + \tau A^T Z A & \Phi_{12} + \tau A^T Z B \\ \Phi_{12}^T + \tau B^T Z A & \Phi_{22} + \tau B^T Z B \end{bmatrix},$$

Φ_{11}, Φ_{12} and Φ_{22} are defined in (8) and Ψ is defined in (9). If $\Xi < 0$ and $\Psi \geq 0$, then $\dot{V}(x_t) < 0$ for any $\xi(t) \neq 0$. Applying the Schur complement (Boyd *et al.*, 1994) shows that Eq. (8) implies $\Xi < 0$. So Σ_0 is asymptotically stable if LMIs (8) and (9) are true. This completes the proof. □

Remark 1: For a time-invariant delay system, according to the procedure of the proof of Theorem 2, it is clear that setting $X_{12} = 0$, $X_{22} = 0$ and $T = 0$ in Theorem 2 yields precisely Theorem 1 in Moon *et al.* (2001). So, Theorem 2 in this paper is an extension of Theorem 1 in Moon *et al.* (2001). Instead of choosing X_{12}, X_{22} and T to be fixed matrices, Theorem 2 selects them by solving LMIs. So, it always chooses suitable ones, thus overcoming the conservatism of Theorem 1 in Moon *et al.* (2001).

Remark 2: If the matrices Y, T and X in Eq. (9) are set to zero, and $Z = \varepsilon I$ (ε is a sufficiently small positive scalar), then Theorem 2 is identical to the well-known delay-independent stability criterion in Gu *et al.* (2003) (Proposition 5.14 on page 169) and Hale & Verduyn Lunel (1993) (Eq. (2.4) on page 134), which is stated as follows:

Corollary 3 *When $\mu = 0$, system Σ_0 is asymptotically stable if there exist real symmetric matrices $P > 0$ and Q such that*

$$\begin{bmatrix} PA + A^T P + Q & PB \\ B^T P & -Q \end{bmatrix} < 0$$

is satisfied.

So, any system that exhibits delay-independent stability, as determined by Corollary 3, is, for all practical purposes, asymptotically stable for any delay satisfying $0 \leq d(t) \leq \tau$, where τ is a positive real number.

Now, extending Theorem 2 to systems with time-varying structured uncertainties yields the following theorem.

Theorem 4 *Given scalars $\tau > 0$ and $\mu < 1$ and assuming (2), the uncertain* *system Σ_1 is robustly stable if there exist symmetric positive definite matrices $P = P^T > 0$, $Q = Q^T > 0$ and $Z = Z^T > 0$, a symmetric semi-positive definite matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$, any matrices Y and T such that LMI (9) and the following LMI are true.*

$$\Omega = \begin{bmatrix} \Phi_{11} + E_a^T E_a & \Phi_{12} + E_a^T E_b & \tau A^T Z & PD \\ \Phi_{12}^T + E_b^T E_a & \Phi_{22} + E_b^T E_b & \tau B^T Z & 0 \\ \tau ZA & \tau ZB & -\tau Z & \tau ZD \\ D^T P & 0 & \tau D^T Z & -I \end{bmatrix} < 0, \tag{13}$$

where Φ_{11}, Φ_{12} and Φ_{22} are defined in (8).

PROOF. Replacing A and B in (8) with $A + DF(t)E_a$ and $B + DF(t)E_b$, respectively, we find that (8) for Σ_1 is equivalent to the following condition.

$$\Phi + \begin{bmatrix} PD \\ 0 \\ \tau ZD \end{bmatrix} F(t) \begin{bmatrix} E_a & E_b & 0 \end{bmatrix} + \begin{bmatrix} E_a^T \\ E_b^T \\ 0 \end{bmatrix} F^T(t) \begin{bmatrix} D^T P & 0 & \tau D^T Z \end{bmatrix} < 0.$$

By Lemma 1, a sufficient condition guaranteeing (8) for Σ_1 is that there exists a positive number $\lambda > 0$ such that

$$\lambda\Phi + \lambda^2 \begin{bmatrix} PD \\ 0 \\ \tau ZD \end{bmatrix} \begin{bmatrix} D^T P & 0 & \tau D^T Z \end{bmatrix} + \begin{bmatrix} E_a^T \\ E_b^T \\ 0 \end{bmatrix} \begin{bmatrix} E_a & E_b & 0 \end{bmatrix} < 0. \tag{14}$$

Replacing $\lambda P, \lambda Q, \lambda Z, \lambda X, \lambda Y$ and λT with P, Q, Z, X, Y and T, respectively, and applying the Schur complement shows that (14) is equivalent to (13). This completes the proof. $\square$

Remark 3: Moon *et al.* (2001) presented an algorithm for constructing a controller with a suboptimal upper bound on the delay based on the method of convex optimization, which stabilizes the system for all admissible uncertainties. The results in this paper combined with that algorithm provide a method of solving the synthesis problem for control systems with a time-varying delay.

4 Numerical Examples

In this section, some examples are used to demonstrate that the method presented in this paper is effective and is an improvement over existing methods.

Example 5 *Consider the uncertain system* Σ_1 *with the following parameters.*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$E_a = \mathrm{diag}\left\{1.6,\ 0.05\right\},\ E_b = \mathrm{diag}\left\{0.1,\ 0.3\right\},\ D = I.$$

This example was given in Kim (2001) and Yue & Won (2002). The upper bounds on the time delay for different μ obtained from Theorem 4 are shown in Table 1. For comparison, the table also lists the upper bounds obtained from the criteria in Li & de Souza (1997), Kim (2001), Yue & Won (2002), Moon *et al.* (2001) and Fridman & Shaked (2002). Note that the results for Fridman & Shaked (2002) were obtained by combining Lemma 1 in their paper with Lemma 1 in this paper. It is clear that Theorem 4 gives much better results than those obtained by Li & de Souza (1997), Kim (2001), Yue & Won (2002) or Moon *et al.* (2001), and the same or better results than Fridman and Shaked's (2002).

Example 6 *Consider the robust stability of the uncertain system* Σ_1 *with the following parameters.*

$$A = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix},$$
$$E_a = E_b = \mathrm{diag}\left\{0.2,\ 0.2\right\},\ D = I.$$

Table 1
Allowable time delay (Example 5).

μ	0	0.5	0.9
Li & de Souza (1997)	0.2013	–	–
Kim (2001)	0.2412	< 0.2	< 0.1
Yue & Won (2002)	0.2412	0.2195	0.1561
Moon *et al.* (2001)	0.7059	–	–
Fridman & Shaked (2002)	1.1490	0.9247	0.6710
Theorem 4	1.1490	0.9247	0.6954

The upper bounds on the time delay obtained from Theorem 4 are listed in Table 2. The results for Fridman & Shaked (2002) in the table were also obtained by combining Lemma 1 in their paper with Lemma 1 in this paper. It is clear that our results are significantly better than those in Fridman & Shaked (2002). In particular, when $\mu = 0.9$, the method in Fridman & Shaked (2002) fails; but the upper bound 0.2420 is obtained by using Theorem 4 in this paper.

Table 2
Allowable time delay (Example 6).

μ	0	0.5	0.9
Fridman & Shaked (2002)	0.6812	0.1820	–
Theorem 4	0.8435	0.2433	0.2420

5 Conclusion

This paper presents a new method of determining delay-dependent stability criteria that takes the relationship between $x(t-d(t))$ and $x(t) - \int_{t-d(t)}^{t} \dot{x}(s)ds$ into account. Some free weighting matrices that express the influence of these two terms are determined based on linear matrix inequalities, which makes it easy to choose suitable ones. It was shown that the criteria in Moon *et al.* (2001) are a special case of this new method, and that the new method is less conservative than existing ones. Finally, some numerical examples suggest that the method presented here is very effective and is a significant improvement over existing ones.

References

Boyd, S., L. EL Ghaoui, Feron, E., & Balakrishnan, V. (1994). Linear matrix inequality in system and control theory. Studies in Applied Mathematics, Philadelphia:SIAM.

Cao, Y.Y., Sun, Y.X., & Cheng, C.W. (1998). Delay-dependent robust stabilization of uncertain systems with multiple state delays. *IEEE Trans. Automat. Contr.*, 43, 1608-1612.

de Souza, C.E., & Li X. (1999). Delay-dependent robust H_∞ control of uncertain linear state-delayed systems. *Automatica*, 35, 1313-1321.

Fridman E. (2001). New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems, *Syst. Contr. Lett.*, 43, 309-319.

Fridman E., & Shaked U. (2002). An improved stabilization method for linear time-delay systems, *IEEE Trans. Automat. Contr.*, 47, 1931-1937.

Fridman E., & Shaked U. (2003). Delay-dependent stability and H_∞ control: constant and time-varying delays, *Int. J. Control*, 76, 48-60.

Gu K., & Niculescu S.I. (2000). Additional dynamics in transformed time delay systems, *IEEE Trans. Automat. Contr.*, 45, 572-575.

Gu K., & Niculescu S.I. (2001). Further remarks on additional dynamics in various model transformations of linear delay systems, *IEEE Trans. Automat. Contr.*, 46, 497-500.

Gu K., Kharitonov V. L., & Chen J. (2003). *Stability of Time-Delay Systems (Control Engineering)*, Springer-Verlag.

Gu, Y., Wang, S., Li, Q., Cheng, Z., & Qian J. (1998). On delay-dependent stability and decay estimate for uncertain systems with time-varying delay. *Automatica*, 34, 1035-1039.

Han, Q.L., & Gu, K.Q. (2001). On robust stability of time-delay systems with norm-bounded uncertainty. *IEEE Trans. Automat. Contr.*, 46, 1426-1431.

Han, Q.L. (2002a). New results for delay-dependent stability of linear systems with time-varying delay. *Int. J. Sys. Sci.*, 33 213-228.

Han, Q.L. (2002b). Robust stability of uncertain delay-differential systems of neutral type. *Automatica*, 38, 719-723.

Hale, J.K. & Verduyn Lunel S.M. (1993). *Introduction to Functional Differential Equations*. New York: Springer-Verlag.

Kim, J.H. (2001). Delay and its time-derivative dependent robust stability of time-delayed linear systems with uncertainty. *IEEE Trans. Automat. Contr.*, 46, 789-792.

Li, X., & de Souza, C.E. (1997). Delay-dependent robust stability and stabilization of uncertain linear delay systems: A Linear Matrix Inequality approach. *IEEE Trans. Automat. Contr.*, 42, 1144-1148.

Moon, Y.S., Park, P., Kwon, W.H., & Lee, Y.S. (2001). Delay-dependent robust stabilization of uncertain state-delayed systems. *Int. J. Control*, 74, 1447-1455.

Park, P. (1999). A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Trans. Automat. Contr.*, 44, 876-877.

Su, T.J., & Huang, C.G. (1992). Robust stability of delay dependence for linear uncertain systems. *IEEE Trans. Automat. Contr.*, 37, 1656-1659.

Xie L. (1996). Output feedback H_∞ control of systems with parameter uncertainty. *Int. J. Control*, 63, 741-750.

Yue, D., & Won, S. (2002). An improvement on 'Delay and its time-derivative dependent robust stability of time-delayed linear systems with uncertainty'. *IEEE Trans. Automat. Contr.*, 47, 407-408.