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SAMPLED-DATA ROBUST CONTROL OF AN ARM ROBOT

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Abstract

This paper presents a design method for a digital tracking control system for a continuous plant with structured uncertainties, which is aimed at the positioning control of an arm robot. To guarantee the robust stability of the closed-loop system and provide the desired closed-loop performance, the design problem is first formulated as a sampled-data $\mathcal{H}_\infty$ control problem, and is then transformed into a discrete-time $\mathcal{H}_\infty$ control problem. Finally, linear matrix inequalities are used to obtain a static state feedback controller and a reduced-order output feedback controller. The validity of the method was demonstrated through simulations and experiments.

1 Introduction

Many mechatronic systems are modeled as a linear continuous time-invariant system (the nominal plant) with some continuous-time uncertainties. Since microcomputer devices are widely used as controllers nowadays, the controller is in a discrete-time form. So, the control system employs two types of signals, namely analog and digital. And the sampled-data system is periodic even if both the plant and the controller are time-invariant. These features make the synthesis of the sampled-data system a difficult task. The usual approach to designing a robust digital control system is first to estimate the equivalent discrete uncertainties, and then to design a robust stabilizing controller for the related discrete uncertain plant. In this regard, over the past few years, sampled-data $\mathcal{H}_\infty$ control, which handles the continuous uncertainties of a plant directly, has provoked a great deal of interest e.g. (**?**), (**?**). Bamieh and Pearson (1992) and Kabamba and Hara (1993).

An arm robot (**?**), (**?**) (Ohyama and Ikebe, 1994; Ohyama *et al.*, 1999) is a typical mechatronic system. Since it is simple, its mathematical model is easy to identify, and the control result (position) can be understood visually, it is widely used in control engineering courses.

This paper considers the problem of designing a robust tracking controller for an arm robot with continuous structured uncertainties. The design problem is first formulated as a sampled-data $\mathcal{H}_\infty$ control problem, and is then transformed into a discrete-time $\mathcal{H}_\infty$ control problem. To reduce the order of an $\mathcal{H}_\infty$ controller, the results in (**?**) Xin *et al.* (1996), in which a reduced-order controller was designed based on linear matrix inequalities (LMI) e.g. (**?**), (**?**) Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994), are used to obtain reduced-order output feedback $\mathcal{H}_\infty$ controllers. The validity of the method was demonstrated through simulations and experiments.

Throughout this paper, $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denotes either the transfer function $G(s) = D + C(sI - A)^{-1}B$ for a continuous-time system, or the pulse transfer function $G(z) = D + C(zI - A)^{-1}B$ for a discrete-time system. $||G(z)||_\infty := \sup_{-\pi<\phi\leq\pi} \bar{\sigma}\left(G(e^{j\phi})\right)$ for $G(z) \in \mathbf{RH}_\infty$, where $\bar{\sigma}(\bullet)$ means the maximum singular value. A continuous-time signal $f(t)$ in $\mathbf{L}_2$ means $\int_0^\infty ||f(t)||^2 dt < \infty$. Parentheses $(\cdot)$ around an independent variable indicate an analog function, while square brackets $[\cdot]$ indicate a discrete sequence. G indicates a continuous-time or discrete-time system, while $\mathcal{G}$ indicates a hybrid system that contains both continuous and discrete-time time-invariant subsystems. $\emptyset$ is the empty set. $A^\perp$ denotes a matrix satisfying $N(A^\perp) = R(A)$ and $A^\perp A^{\perp T} > 0$ with $N(A)$ and $R(A)$ denoting the null space and the range space of matrix A, respectively.

2 System Description and Problem Formulation

The plant to be controlled in this study is the arm robot shown in Fig. 1. The arm is connected to a motor through a gear box. The position of the arm is detected by an encoder mounted at the axis of the motor. A block diagram of the plant is drawn in Fig.

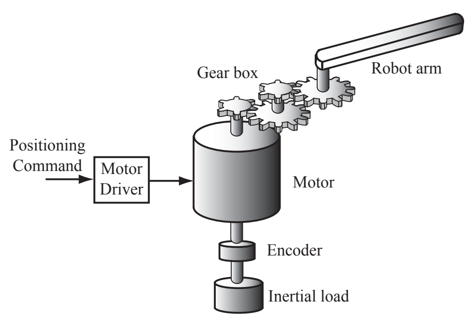

Figure 1: Arm robot.

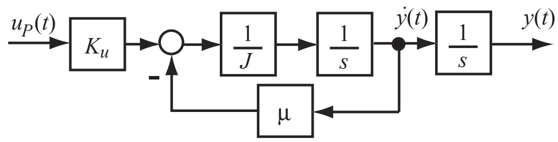

Figure 2: Block diagram of the plant.

2. Its mathematical model is described by

$$\begin{cases} \dot{x}_P(t) = (A_P + \Phi\Gamma(t)\Psi_A)x_P(t) + (B_P + \Phi\Gamma(t)\Psi_B)u_P(t), \\ y(t) = C_P x_P(t), \\ y_F(t) = C_F x_P(t), \end{cases} \tag{1}$$

where

$$\begin{cases} \bar{\alpha} = (\alpha_M + \alpha_m)/2; \qquad \delta_\alpha = (\alpha_M - \alpha_m)/2, \\ \bar{\beta} = (\beta_M + \beta_m)/2; \qquad \delta_\beta = (\beta_M - \beta_m)/2, \\ A_P = \begin{bmatrix} 0 & 1 \\ 0 & -\bar{\alpha} \end{bmatrix}; \qquad B_P = \begin{bmatrix} 0 \\ \bar{\beta} \end{bmatrix}; \quad C_P = [\,1 \quad 0\,], \\ \Phi = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}; \qquad \Gamma(t) = \begin{bmatrix} (\beta - \bar{\beta})/\delta_\beta & 0 \\ 0 & (\alpha - \bar{\alpha})/\delta_\alpha \end{bmatrix}, \\ \Psi_A = \begin{bmatrix} -\delta_\beta & 0 \\ 0 & -\delta_\alpha \end{bmatrix}; \quad \Psi_B = \begin{bmatrix} \delta_\beta \\ 0 \end{bmatrix}, \end{cases}$$

and $y(t)$, $x_P(t) = [\,x_{P1}(t) \quad x_{P2}(t)\,]^T := [\,y(t) \quad \dot{y}(t)\,]^T \in \mathbf{R}^{n_P}$, $u_P(t)$ and $y_F(t) \in \mathbf{R}^{m_P}$ are the position of the arm, the state of the plant, the voltage command and the observed output, respectively. In particular, $C_F = I_{n_P}$ means the state feedback, and $C_F = C_P$ means the output feedback. Without loss of generality, $C_P = [\ c_{P1} \quad 0\ ]$, $c_{P1} \neq 0$ ($c_{P1} \in \mathbf{R}$) is assumed. $\alpha = \mu/J$, and $\beta = K_u/J$. J, μ and K_u are the inertia of the system, the friction coefficient, and the torque coefficient, respectively. $\alpha \in [\,\alpha_m \quad \alpha_M\,]$ and $\beta \in [\,\beta_m \quad \beta_M\,]$ reflect the variations in the inertial load etc. $\Gamma(t)$ is an unknown bounded matrix ($\Gamma^T(t)\Gamma(t) \leq I$) that represents the time-varying parameter uncertainties.

For this plant, the robust tracking control system is constructed as shown in Fig. 3.

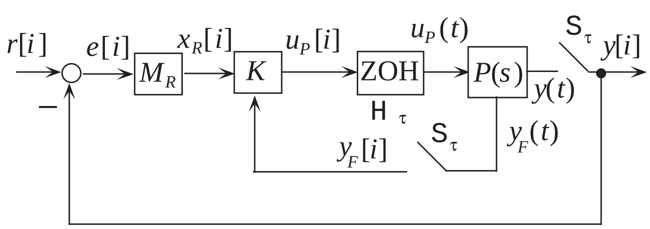

Figure 3: Configuration of robust tracking control system.

Let the reference input be

$$\begin{cases} r(z) = \dfrac{\bar{r}(z)}{\phi_R(z)}; \\ \bar{r}(z) = r_0 z^L + r_1 z^{L-1} + \ldots + r_{L-1} z + r_L; \\ \phi_R(z) = z^L + \phi_1 z^{L-1} + \ldots + \phi_L \end{cases} \tag{2}$$

with all the roots of $\phi_R(z) = 0$ being outside of the open unit circle, where $1/\phi_R(z)$ is the generator of the reference input and $\bar{r}(z)$ is the initial function. Then, the state space representation of the internal model of the reference input, $M_R(z)$, is

$$\begin{cases} x_R[i+1] = A_R x_R[i] + B_R e_R[i], \\ A_R = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -\phi_L & -\phi_{L-1} & \ldots & -\phi_1 \end{bmatrix} \in \mathbf{R}^{L\times L}, \\ B_R = [0 \quad \ldots \quad 0 \quad 1]^T \in \mathbf{R}^{L\times 1}. \end{cases} \tag{3}$$

This paper considers the design of a reduced-order discrete-time controller $K(z)$ ($u_P[i] = K \begin{bmatrix} y_F[i] \\ x_R[i] \end{bmatrix}$) with an order less than n_P (the order of the plant), that robustly stabilizes the control system and tracks a given reference input without steady-state error.

3 Design of Discrete-time Controller

Redrawing Fig. 3 with $r[i] = 0$ gives Fig. 4, in which the two new signals $v(t)$ and $w(t)$ are defined to be the input and output of the uncertainty $\Gamma(t)$, respectively; and the other new signals, $v_u(t)$, $v_P(t)$ and $v_R[i]$, are the control input, and the states of the plant and internal model weighted by positive semi-definite matrices $Q_u^{1/2}$, $Q_P^{1/2}$ and $Q_R^{1/2}$, respectively.

Applying the small gain theorem of a sampled-data system (**?**) to the control system yields the following condition for robust stability.

$$\|\mathcal{G}_{\mathcal{P}}\|_\infty := \sup_{w(t)\in \mathbf{L}_2} \frac{\|v(t)\|_2}{\|w(t)\|_2} < 1. \tag{4}$$

To guarantee robust stability and obtain the desired closed-loop performance, we extend the controlled output to include $v_u(t)$, $v_P(t)$ and $v_R[i]$, which are the control input, and the states of the plant and internal model weighted by positive semi-definite matrices $Q_u^{1/2}$, $Q_P^{1/2}$ and $Q_R^{1/2}$, respectively. Let $v_a := \begin{bmatrix} v(t) & v_u(t) & v_P(t) & v_R[i] \end{bmatrix}^T$, then the design problem for the robust controller can be formulated as:

Find a reduced-order controller $K(z)$ that internally stabilizes the generalized plant $\mathcal{P}_\mathcal{S}$ described by

$$\begin{bmatrix} v_a \\ \hdashline y_F[i] \\ x_R[i] \end{bmatrix} = \mathcal{P}_\mathcal{S} \begin{bmatrix} w(t) \\ \hdashline u_P[i] \end{bmatrix} \tag{5}$$

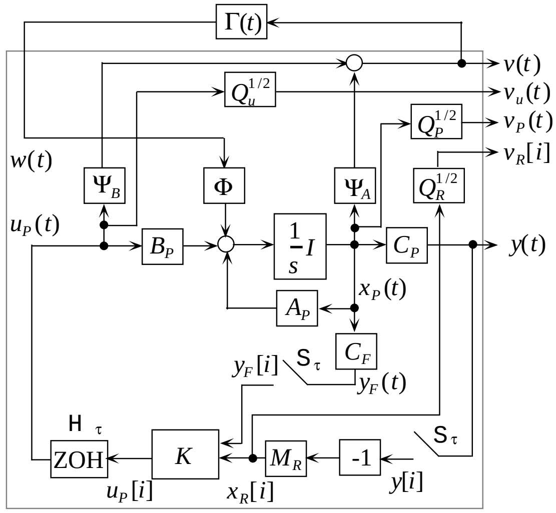

Figure 4: Design of robust controller.

and satisfies $\|\mathcal{G}_{\mathcal{P}_\mathcal{S}}\|_\infty < 1$, *where* $\mathcal{G}_{\mathcal{P}\mathcal{S}} = \mathcal{P}_\mathcal{S} * K = \mathcal{P}_{\mathcal{S}11} + \mathcal{P}_{\mathcal{S}12} K (I - \mathcal{P}_{\mathcal{S}22} K)^{-1} \mathcal{P}_{\mathcal{S}21}$ *is the linear fraction transformation (LFT) of* $\mathcal{P}_\mathcal{S}$ *and* K, *and* $\mathcal{P}_\mathcal{S}$ *is given by*

$$\mathcal{P}_\mathcal{S} = \begin{bmatrix} \mathcal{P}_{\mathcal{S}11} & \mathcal{P}_{\mathcal{S}12} \\ \mathcal{P}_{\mathcal{S}21} & \mathcal{P}_{\mathcal{S}22} \end{bmatrix} = \left[\begin{array}{cc|c:c} A_R & -B_R \mathcal{S}_\tau C_P & 0 & 0 \\ 0 & A_P & \Phi & B_P \mathcal{H}_\tau \\ \hline 0 & \Psi_A & 0 & \Psi_B \mathcal{H}_\tau \\ 0 & 0 & 0 & Q_u^{1/2} \mathcal{H}_\tau \\ 0 & Q_P^{1/2} & 0 & 0 \\ Q_R^{1/2} & 0 & 0 & 0 \\ \hdashline 0 & \mathcal{S}_\tau C_F & 0 & 0 \\ I_L & 0 & 0 & 0 \end{array}\right]. \qquad (6)$$

Note that the generalized plant $\mathcal{P}_\mathcal{S}$ contains both continuous and discrete parts. We first convert the design problem to an equivalent discrete-time $\mathcal{H}_\infty$ control problem by the following steps:

Step 1 Partition the generalized plant $\mathcal{P}_\mathcal{S}$ in Fig. 4 into two sub-systems: a continuous subsystem, $P_C(s)$, and a discrete subsystem, $P_D(\lambda)$, as shown in Fig. 5.

Step 2 Lift the continuous sub-system

$$P_C(s) : \left[\begin{array}{c:c} w(t) & u_P(t) \end{array}\right]^T \rightarrow$$

$$\left[\begin{array}{ccc:cc} v(t) & v_u(t) & v_P(t) & y(t) & y_F(t) \end{array}\right]^T$$

with its state space realization being

$$P_C(s) = \left[\begin{array}{c|c:c} A_P & \Phi & B_P \\ \hline \Psi_A & 0 & \Psi_B \\ 0 & 0 & Q_u^{1/2} \\ Q_P^{1/2} & 0 & 0 \\ \hdashline C_P & 0 & 0 \\ C_F & 0 & 0 \end{array}\right]$$

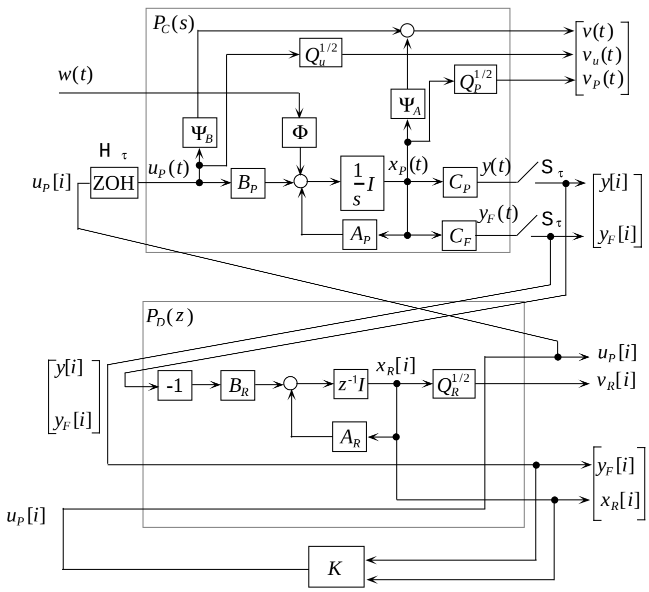

Figure 5: Partitioning of system.

and obtain the equivalent finite-dimensional discrete-time time-invariant system

$$\check{P}_C(\lambda) = \left[\begin{array}{c|c:c} A_c & B_{c1} & B_{c2} \\ \hline C_c & D_{c11} & D_{c12} \\ \hdashline C_P & 0 & 0 \\ C_F & 0 & 0 \end{array}\right]. \tag{7}$$

Step 3 Combine Eq. (7) with the discrete sub-system

$$P_D(\lambda) : \left[\begin{array}{cc:c} y[i] & y_F[i] & u_P[i] \end{array}\right]^T \to \left[\begin{array}{cc:cc} u_P[i] & v_R[i] & y_F[i] & x_R[i] \end{array}\right]^T;$$

$$P_D(\lambda) = \left[\begin{array}{c|cc:c} A_R & -B_R & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ Q_R^{1/2} & 0 & 0 & 0 \\ \hdashline 0 & 0 & I & 0 \\ I_L & 0 & 0 & 0 \end{array}\right] \tag{8}$$

using an LFT to obtain the equivalent generalized plant $P_e(\lambda) = \check{P}_C(\lambda) * P_D(\lambda)$:

$$P_e(\lambda) := \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c:c} A_R & -B_R C_P & 0 & 0 \\ 0 & A_c & B_{c1} & B_{c2} \\ \hline 0 & C_c & D_{c11} & D_{c12} \\ Q_R^{1/2} & 0 & 0 & 0 \\ \hdashline 0 & C_F & 0 & 0 \\ I_L & 0 & 0 & 0 \end{array}\right]. \tag{9}$$

Now, the design problem can be stated as: *Find a reduced-order controller $K(z)$ that internally stabilizes the equivalent discrete-time generalized plant $P_e(z)$ and satisfies*

$$\|G_{P_e}(z)\|_\infty < 1.$$

where $G_{P_e}(z) = P_e(z) * K(z)$ is the LFT of $P_e(z)$ and $K(z)$.

It is known from (**?**) that the $\mathcal{H}_\infty$ control problem for the discrete-time system is solvable if and only if $\mathcal{L}_D \neq \emptyset$ where

$$\mathcal{L}_D := \left\{(X,Y) : X \in \mathcal{L}_B, Y \in \mathcal{L}_C, \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0\right\}, \tag{10}$$

$$\mathcal{L}_B := \left\{X : X = X^T > 0, \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^\perp M_B \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^{\perp T} < 0\right\}, \tag{11}$$

$$\mathcal{L}_C := \left\{Y : Y = Y^T > 0, \begin{bmatrix} C_2^T \\ 0 \end{bmatrix}^\perp M_C \begin{bmatrix} C_2^T \\ 0 \end{bmatrix}^{\perp T} < 0\right\}, \tag{12}$$

$$M_B := \begin{bmatrix} AXA^T - X + B_1B_1^T & AXC_1^T + B_1D_{11}^T \\ C_1XA^T + D_{11}B_1^T & C_1XC_1^T + D_{11}D_{11}^T - I \end{bmatrix}, \tag{13}$$

$$M_C := \begin{bmatrix} A^TYA - Y + C_1^TC_1 & A^TYB_1 + C_1^TD_{11} \\ B_1^TYA + D_{11}^TC_1 & B_1^TYB_1 + D_{11}^TD_{11} - I \end{bmatrix}. \tag{14}$$

Suppose $\mathcal{L}_D \neq \emptyset$. Then, there exists an $\mathcal{H}_\infty$ controller of order n_d satisfying

$$n_d \leq \text{rank}\,(Y - X^{-1}). \tag{15}$$

If the above result were directly used to find an $\mathcal{H}_\infty$ controller, the order of the feedback controller would generally be $n_L = L + n_P$. Since x_R described by Eq. (3) is known, in the rest of this section, the design of reduced-order controllers for two special cases – state feedback and output feedback – is considered. Applying the results of Xin *et al.* (1996) gives the following results.

THEOREM 1 *Suppose the discrete-time $\mathcal{H}_\infty$ control problem for the generalized plant (9) with $C_F = C_P$ is solvable. Let the LMI solution be $(X,Y) \in \mathcal{L}_D$, with $\mathcal{L}_D$ being defined in Eq. 10. Decompose $Y := \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$, where $Y_{11} \in \mathbf{R}^{(L+1)\times(L+1)}$ and $Y_{22} \in \mathbf{R}^{(n_P-1)\times(n_P-1)}$, according to*

$$C_2 = \begin{bmatrix} 0 & C_P \\ I_L & 0 \end{bmatrix} = \left[\begin{array}{cc:c} 0 & c_{P1} & 0 \\ I_L & 0 & 0 \end{array}\right] := \left[\begin{array}{c:c} C_{21} & 0 \end{array}\right], \tag{16}$$

and also decompose $Z := Y - X^{-1} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}$. Let

$$\bar{Y}_{11} := Y_{11} - Z_{11} + Z_{12}Z_{22}^+Z_{12}^T, \tag{17}$$

and construct

$$\bar{Y} := \begin{bmatrix} \bar{Y}_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}.$$

Then,

$$(X, \bar{Y}) \in \mathcal{L}_D, \tag{18}$$

$$rank(\bar{Y} - X^{-1}) = rank Z_{22} \le n_P - 1 \tag{19}$$

holds, which imply that a feedback controller, $K_2(z)$, with an order less than or equal to $n_P - 1$ can be constructed by applying the standard LMI algorithm to $(X, \bar{Y})$.

Proof In accordance with the decomposition of C_2, we decompose $P_e(\lambda)$ in (9) into

$$P_e(z) = \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{21} \\ A_{21} & A_{22} & B_{12} & B_{22} \\ \hline C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & 0 & D_{21} & 0 \end{array} \right]. \tag{20}$$

Since C_{21} defined in Eq. (16) is invertible,

$$\left[\begin{array}{c} C_2^T \\ D_{21}^T \end{array} \right]^{\perp} = \left[\begin{array}{cc} 0 & I \end{array} \right]. \tag{21}$$

Therefore, based on the decomposition of Y, writing out $\mathcal{L}_c$ in Eq. (12) gives us

$$\left[\begin{array}{cc} 0 & I \end{array} \right] \left[\begin{array}{cc} * & * \\ * & L(Y) \end{array} \right] \left[\begin{array}{c} 0 \\ I \end{array} \right] < 0,$$

where

$$L(Y) := \left[\begin{array}{cc} L_{11}(Y) & L_{12}(Y) \\ L_{12}^T(Y) & L_{22}(Y) \end{array} \right],$$

with its detail being omitted for brevity. Thus, $L(Y) < 0$ holds. From (17), we have $\bar{Y}_{11} \le Y_{11}$, which yields

$$L(\bar{Y}) - L(Y) = \left[\begin{array}{c} A_{12}^T \\ B_{11}^T \end{array} \right] (\bar{Y}_{11} - Y_{11}) \left[\begin{array}{cc} A_{12} & B_{11} \end{array} \right] \le 0.$$

Therefore, $L(\bar{Y}) < 0$. From Eq. (17), we obtain

$$\bar{Y} - X^{-1} = \left[\begin{array}{c} Z_{12} Z_{22}^{+} \\ I \end{array} \right] Z_{22} \left[\begin{array}{cc} Z_{22}^{+} Z_{12}^T & I \end{array} \right] \ge 0. \tag{22}$$

Thus, $(X, \bar{Y}) \in \mathcal{L}_D$. Eq. (22) gives Eq. (19).

Similar to Theorem 1, we can obtain

COROLLARY 1 *The discrete-time $\mathcal{H}_\infty$ control problem for the generalized plant (9) with $C_F = I_{n_P}$ is solvable if and only if $\mathcal{L}_D \neq \emptyset$, where $\mathcal{L}_D$ is defined in Lemma* **??**, *with $\mathcal{L}_C$ being simplified to*

$$\mathcal{L}_C := \{Y : Y = Y^T > 0, B_1^T Y B_1 + D_{11}^T D_{11} - I < 0\}. \tag{23}$$

If it is solvable with the LMI solution $(X_0, Y_0) \in \mathcal{L}_D$, then $(X_0, X_0^{-1}) \in \mathcal{L}_D$ holds, from which it follows that there exists a static state feedback controller.

Based on Theorem 1 and Corollary 1, the reduced-order output feedback controller and static state feedback controller can easily be obtained by using the Sampled-data Control Toolbox (**?**) (Fujioka *et al.*, 1999) and LMI Toolbox (**?**) (Gahinet *et al.*, 1995) of MATLAB.

Summarizing the above results gives the design procedure for the sampled-data robust tracking control. It is divided into steps as follows.

Step 1: Calculate the plant, Eq. (1), and the reference input, Eq. (2).

Step 2: Choose the semi-definite weighting matrices $Q_u^{1/2}$, $Q_P^{1/2}$ and $Q_R^{1/2}$, and construct the generalized plant, $\mathcal{P}_{\mathcal{S}}$ (Eq. (6)).

Step 3: Use Steps 1 - 3 in Section 3 to convert the sampled-date $\mathcal{H}_\infty$ control problem to an equivalent discrete-time $\mathcal{H}_\infty$ control problem.

Step 4: Use the standard LMI algorithm (Equations (10), (11) and (12)) to calculate a feedback controller, $K(z)$.

Step 5: Use Corollary 1/Theorem 1 to obtain a reduced-order state/output feedback controller, $K(z)$.

4 Simulation and experimental results

The experimental system is shown in Fig. 6. The arm was driven by a DC motor (rated voltage: $3\,\mathrm{V}$; rated current: $0.56\,\mathrm{A}$; rated speed: $895\,\mathrm{r}ad/s$). An A4-size notebook computer (700-MHz Celeron) was used for control. A motor driver, a counter and a D/A converter were built into the interface box. A parallel connection was used between the interface box and the conputer. The rotational speed was reduced by a gear box (64.8:1) and an optical encoder (16 cycles per turn) was mounted on the shaft of the motor to measure the angle of the arm. So, the resolution for the arm is $6.06 \times 10^{-3}\,\mathrm{r}ad/pulse$. Pulses from the encoder were sent to the counter in the interface box. The control input was fed to the motor through the interface box. The parameters of the plant without an additional inertial load are

$$\alpha_M = 8.33; \quad \beta_M = 44.2. \tag{24}$$

We built some brass rods to simulate a change in the inertial load. When the heaviest one (diameter: $15\,\mathrm{m}m$; length: $10\,\mathrm{m}m$) is mounted on the shaft of the motor, the parameters of the plant are

$$\alpha_m = 4.17; \quad \beta_m = 22.1. \tag{25}$$

Figure 6: Photograph of the experimental system.

So, for the experimental system, the parameters α and β take the following values

$$\alpha \in [4.17, 8.33]; \quad \beta \in [22.1, 44.2]. \tag{26}$$

The nominal plant is

$$\bar{\alpha} = 6.25, \quad \bar{\beta} = 33.2. \tag{27}$$

We designed tracking controllers that robustly stabilize the control system for which the output tracks the reference input

$$r(t) = 1(t) \tag{28}$$

without steady-state error at the sampling points. The sampling period was chosen to be

$$\tau = 0.05\,\mathrm{s}.$$

So, the internal model of the reference input is given by

$$\phi_R(z) = z - 1. \tag{29}$$

A state feedback controller was designed under the conditions

$$Q_u^{1/2} = 1, \quad Q_P^{1/2} = \left(C_P^T C_P\right)^{1/2} = diag\{1, 0\}, \tag{30}$$

and

$$Q_R^{1/2} = 0.65 \tag{31}$$

for the state feedback case $C_F = I_2$; and

$$Q_R^{1/2} = 0.75 \tag{32}$$

for the output feedback case $C_F = C_P$. Corollary 1 yields a *static* state feedback controller

$$K = \begin{bmatrix} 2.669 & -94.93 & -6.088 \end{bmatrix}. \tag{33}$$

And an output feedback controller is designed using Theorem 1. It is of the following form

$$K = \left[\begin{array}{c|cc} -1.650 \times 10^{-1} & -5.206 & 6.452 \times 10^{-4} \\ \hline -1.669 \times 10^{2} & -8.033 \times 10^{2} & 2.019 \end{array}\right] \tag{34}$$

It has an order of *one* ($n_P - 1 = 1$).

The simulation results are shown in Figs. 7 - 8. It can be seen that the system is stable when the inertial load changes from zero to the heaviest one, and the output tracks the reference input without steady-state error.

We also carried out experiments using the designed controllers. For example, Figs. 9 - 10 shows the experimental results for the heaviest inertial load. As was seen in the simulation results, the robust stability resulting from the sampled-data $\mathcal{H}_\infty$ control was demonstrated. On the other hand, The voltage applied to the motor reached saturation at $\pm 5\,\mathrm{V}$, and the influence of the static friction and dead zone was marked. For these reasons, the response was not as good as the simulations.

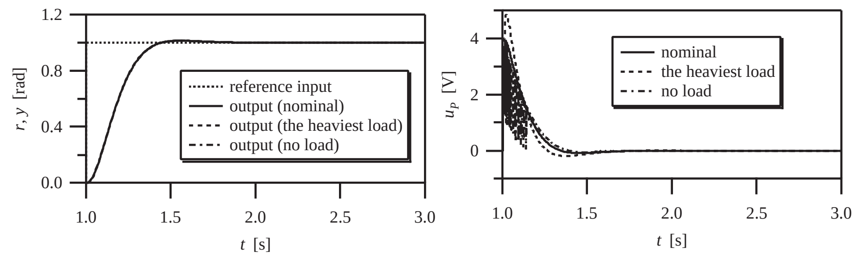

Figure 7: Simulation results of the state feedback case.

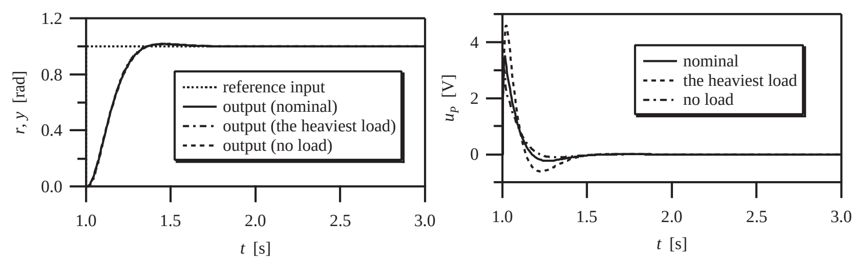

Figure 8: Simulation results of the output feedback case.

5 Conclusions

This paper describes a design method for digital tracking control systems for a continuous plant with structured uncertainties. The design problem is first formulated as a sampled-data $\mathcal{H}_\infty$ control problem, and then transformed into an equivalent discrete-time $\mathcal{H}_\infty$ control problem. A reduced-order output feedback controller with an order no greater than that of the plant minus one have been designed by using an LMI-based $\mathcal{H}_\infty$ control approach. The design method was applied to an arm robot, and the validity of the method has been demonstrated through simulations and experiments.

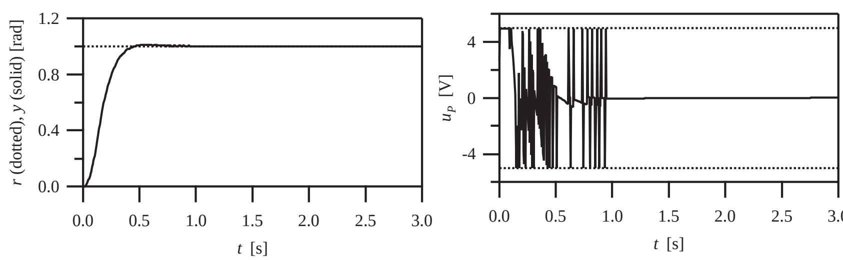

Figure 9: Experimental results for the heaviest inertial load (state feedback).

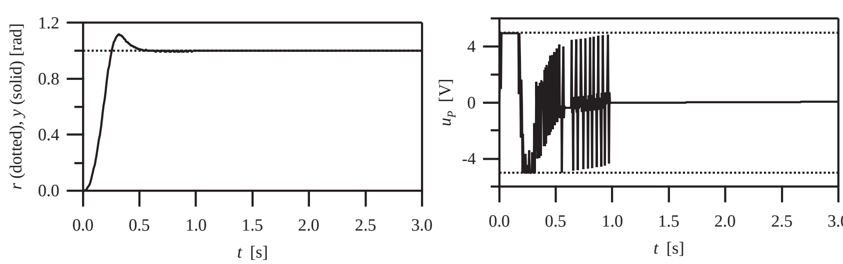

Figure 10: Experimental results for the heaviest inertial load (output feedback).