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Stability Analysis and Control Law Design for Acrobots

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***Abstract*— This paper presents a novel control strategy based on a non-smooth Lyapunov function to guarantee the stability of the system in the whole motion space. Three control laws that are derived based on three Lyapunov functions are applied successively in three stages of motion control to achieve a suitable posture and increase the energy so as to make the acrobot move into the area around the unstable inverted equilibrium position, and to stabilize it at that position. These three Lyapunov functions are combined into one non-smooth function, which theoretically guarantees the stability of the acrobot in the whole motion space. Simulation results have demonstrated the validity of this strategy.**

I. Introduction

An acrobot is a two-link manipulator operating in a vertical plane with an actuator at the elbow but no actuator at the shoulder [1]. A common control objective is to drive the acrobot away from the stable downward equilibrium position and balance it at the unstable inverted equilibrium position. In order to simplify the design procedure, the motion space is usually divided into two subspaces: the attractive area, which is the area around the unstable inverted equilibrium position; and the swing-up area, which is the rest. For example, Spong [1] described a method based on partial feedback linearization to swing an acrobot up, and used a linear quadratic regulator (LQR) to balance it. However, the control laws for LQR balancing are only applicable within a small region [1]. This limitation makes it difficult to capture the acrobot and restrict it to the attractive area. On the other hand, the control methods in [2], [3] are complicated and/or have a relatively long settling time. A fuzzy control strategy was presented in [4] that combines model-free and model-based fuzzy controllers to control the swing-up and balancing motions. This strategy is very simple, and the control results are quite satisfactory. However, it does not solve the combined problem of how to ensure a suitable posture and energy. Thus, it does not theoretically guarantee that the acrobot smoothly enters the attractive area and stays there.

Åström and Furuta proposed a method to swing up a pendulum based on energy control [5], and Fantoni *et al.* derived an energy-based control strategy for a pendubot [6]. The energy-based method has also been used to an acrobot and examined the combined problem of how to ensure a suitable posture and energy for an acrobot [7], [8]. While [6] analyzed the stability in the swing-up area, other studies considered the stability in the attractive area. However, few have considered the stability of the system in the whole motion space. Since an acrobot is a very complex, nonlinear system, it is not easy to design a control law that always provides a suitable combination of posture and energy that makes the acrobot move smoothly into the attractive area and stay there, and also guarantees the stability of the system in the whole motion space. This motivated us to find a more efficient control method and examine the issue of stability in the whole motion space.

This paper presents a control strategy that employs a non-smooth Lyapunov function to handle both swing-up and balancing control, and that guarantees the stability of the system in the whole motion space. First, three control laws are derived based on three Lyapunov functions. They are applied successively in three stages of motion control. In the first stage, the first law changes the posture and energy of the acrobot so as to make the energy increase while the second link stretches out in a natural way. This law makes it possible for the acrobot to quickly approach the attractive area. When its energy reaches a certain level, the acrobot enters the second stage and the second control law takes over to avoid the singularity produced by the first one. This law forces both the angle and the angular velocity of the second link toward zero. In addition, fuzzy rules are employed to adjust the parameters of the control law so as to prevent an abrupt change in energy, to increase the energy toward the potential energy that the acrobot has at the unstable inverted equilibrium position, and keep it near that value. The third stage is reached when the acrobot enters the attractive area

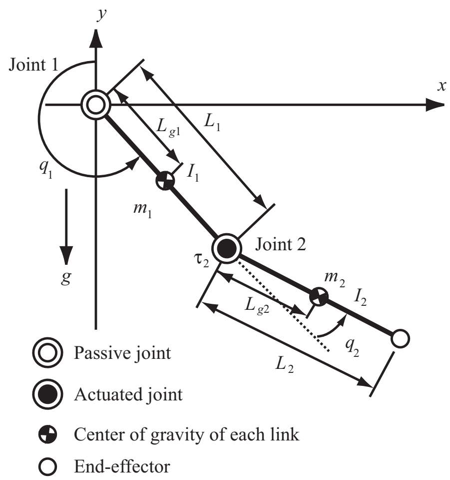

Fig. 1. Model of an acrobot.

and the third control law takes over. This law is based on a linear approximate model for the unstable inverted equilibrium position, and it balances the acrobot in the attractive area. These three Lyapunov functions are combined into one non-smooth function, which theoretically guarantees the stability of the acrobot in the whole motion space. Simulation results have demonstrated the validity of this strategy.

II. Design of Controllers

This section presents the mathematical model of an acrobot and explains the design of the three control laws.

A. Dynamics of acrobot

The acrobot is shown in Fig. 1. If we let q_i be the angle of the i-th link ($i = 1, 2$), $q := [q_1\ q_2]^T$, $\dot{q} := dq/dt$, and $\ddot{q} := d^2q/dt^2$, then the dynamics of the acrobot are given by

$$M(q)\ddot{q} + D(q,\dot{q}) + G(q) = \tau, \tag{1}$$

where

$$\begin{aligned}
M(q) &= \begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix}, && (2)\\
D(q,\dot{q}) &= \left[d_1(q,\dot{q})\ d_2(q,\dot{q})\right]^T, && (3)\\
G(q) &= \left[g_1(q)\ g_2(q)\right]^T. && (4)\\
\tau &= \left[0\ \tau_2\right]^T. && (5)
\end{aligned}$$

The components of $M(q)$, $D(q,\dot{q})$, and $G(q)$ are

$$\begin{aligned}
m_{11}(q) &= m_1 L_{g1}^2 + I_1 + m_2 L_{g2}^2 + I_2 \\
&\quad +2m_2 L_1 L_{g2}\cos q_2 + m_2 L_1^2, \\
m_{12}(q) &= m_{21}(q) = m_2 L_{g2}^2 + I_2 + m_2 L_1 L_{g2}\cos q_2, \\
m_{22}(q) &= m_2 L_{g2}^2 + I_2, \\
d_1(q,\dot{q}) &= -m_2 L_1 L_{g2}\left(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2\right)\sin q_2, \\
d_2(q,\dot{q}) &= m_2 L_1 L_{g2}\dot{q}_1^2 \sin q_2, \\
g_1(q) &= -(m_1 L_{g1} + m_2 L_1) g \sin q_1 \\
&\quad -m_2 L_{g2} g \sin(q_1+q_2), \\
g_2(q) &= -m_2 L_{g2} g \sin(q_1+q_2),
\end{aligned}$$

where, for the i-th link ($i = 1, 2$), m_i is the mass, L_i is the length, L_{gi} is the length from the mass center to the i-th joint, and I_i is the moment of inertia around the center of gravity.

Defining $x = [x_1\ x_2\ x_3\ x_4]^T = [q_1\ q_2\ \dot{q}_1\ \dot{q}_2]^T$ and rewriting the dynamics (1) in the state-space form yields

$$\begin{aligned}
\dot{x}_1 &= x_3, && (6)\\
\dot{x}_2 &= x_4, && (7)\\
\dot{x}_3 &= f_\mu(x) + b_\mu(x)\tau_2, && (8)\\
\dot{x}_4 &= f_\eta(x) + b_\eta(x)\tau_2, && (9)
\end{aligned}$$

where $f_\mu(x)$, $b_\mu(x)$, $f_\eta(x)$, and $b_\eta(x)$ are nonlinear functions given by

$$\begin{aligned}
\begin{bmatrix} f_\mu(x) \\ f_\eta(x) \end{bmatrix} &= M^{-1}(q)\begin{bmatrix} -d_1(q,\dot{q}) - g_1(q) \\ -d_2(q,\dot{q}) - g_2(q) \end{bmatrix}, && (10)\\
\begin{bmatrix} b_\mu(x) \\ b_\eta(x) \end{bmatrix} &= M^{-1}(q)\begin{bmatrix} 0 \\ 1 \end{bmatrix}. && (11)
\end{aligned}$$

The system (6)-(9) can be written as

$$\dot{x} = f(x) + b(x)\tau_2, \tag{12}$$

where

$$\begin{aligned}
f(x) &= \left[x_3\ x_4\ f_\mu(x)\ f_\eta(x)\right]^T, \\
b(x) &= \left[0\quad 0\quad b_\mu(x)\quad b_\eta(x)\right]^T.
\end{aligned}$$

B. Division of motion space

The total mechanical energy, $E(x)$, of the acrobot is the sum of the kinetic energy, $T(x)$, and the potential energy, $V(x)$:

$$E(x) = T(x) + V(x). \tag{13}$$

$T(x)$ and $V(x)$ are expressed in generalized coordinates as

$$\begin{aligned}
T(x) &= \frac{1}{2}\left[x_3\quad x_4\right] M(x)\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, && (14)\\
V(x) &= \sum_{i=1}^{2} V_i(x) = \sum_{i=1}^{2} m_i g h_i(x), && (15)
\end{aligned}$$

where $V_i(x)$ is the potential energy and $h_i(x)$ is the height of the center of mass of the i-th link ($i = 1, 2$). $h_i(x)$ ($i = 1, 2$) are given by

$$\begin{aligned}
h_1(x) &= L_{g1}\cos x_1, \\
h_2(x) &= L_1\cos x_1 + L_{g2}\cos(x_1+x_2).
\end{aligned}$$

Let the whole motion space be Σ, and define the attractive area Σ_a to be

$$\Sigma_a : \begin{cases} \left|\bmod\left(\dfrac{x_1}{2\pi}\right)\right| \le \beta_1, \\ \left|\bmod\left(\dfrac{x_1+x_2}{2\pi}\right)\right| \le \beta_2, \\ |E(x) - E_0| \le \epsilon, \end{cases} \tag{16}$$

where mod$(\cdot)$ means the residue modulus 2π. E_0 is the potential energy at the unstable inverted equilibrium position; and β_1, β_2 and ϵ are positive real numbers. So, the swing-up area is $\Sigma - \Sigma_a$.

C. First control law

The first Lyapunov function, $J_1(x)$, is chosen to be

$$J_1(x) = \frac{1}{2}\left\{k_{p1}x_2^2 + k_{d1}x_4^2 + k_{e1}\left[E(x) - E_0\right]^2\right\} + \Delta_1, \tag{17}$$

where $k_{p1} > 0$, $k_{d1} > 0$ and $k_{e1} > 0$ are constants; and $\Delta_1 \geq 0$ is a constant that plays a key role in guaranteeing that the non-smooth Lyapunov function discussed in the next section decreases monotonically. The choice of Δ_1 is also explained in that section.

As shown in [4], the derivative of the energy with respect to time is

$$\dot{E}(x) = x_4\tau_2. \tag{18}$$

Applying (9) yields the derivative of $J_1(x)$ with respect to time

$$\begin{aligned} \dot{J}_1(x) &= x_4\left\{k_{p1}x_2 + k_{d1}\dot{x}_4 + k_{e1}\left[E(x) - E_0\right]\tau_2\right\} \\ &= x_4\left\{k_{p1}x_2 + k_{d1}f_\eta(x) + \left[k_{d1}b_\eta(x) + k_{e1}\left(E(x) - E_0\right)\right]\tau_2\right\}. \end{aligned} \tag{19}$$

So, the first control law is chosen to be

$$C_1:\ \tau_2 = -\frac{k_{p1}x_2 + k_{d1}f_\eta(x) + \lambda_1 \mathrm{sat}(x_4/\Phi_1)}{k_{d1}b_\eta(x) + k_{e1}(E(x) - E_0)}, \tag{20}$$

where $\lambda_1 > 0$ and $\Phi_1 > 0$ are constants.

This law ensures that

$$\dot{J}_1(x) = -\lambda_1 x_4 \mathrm{sat}(x_4/\Phi_1) \leq 0. \tag{21}$$

It will be shown in the next section that, when x is in the swing-up area, (21) guarantees that the energy increases and, at the same time, that the second link straightens out in a natural way.

Note that, if (20) is applied while $E(x) < E_0$, then a singularity occurs when

$$E(x) = E_0 - \frac{k_{d1}}{k_{e1}}b_\eta(x), \tag{22}$$

In this paper, a second control law is employed to avoid this situation, and the first switching condition is

$$T_1: \left\{\begin{array}{l} J_2(x) < J_1(x), \\ k_{d1}b_\eta(x) + k_{e1}\left[E(x) - E_0\right] \leq \zeta, \end{array}\right. \tag{23}$$

where $J_2(x)$ is the second Lyapunov function, which is explained in the next section, and $\zeta < 0$ is a constant.

This condition divides the swing-up area into two subspaces: Σ_{s1} and Σ_{s2}. When (23) is satisfied, the control law switches from (20), which is used in Σ_{s1}, to the second one, which is used in Σ_{s2}, as described below.

D. Second control law

The second Lyapunov function, $J_2(x)$, is defined to be

$$J_2(x) = \frac{1}{2}\left(k_{p2}x_2^2 + k_{d2}x_4^2\right) + \Delta_1, \tag{24}$$

where $k_{p2} > 0$ and $k_{d2} > 0$ are constants.

The angle, x_2, and the angular velocity, x_4, of the second link approach zero if $\dot{J}_2(x)$ is negative. Equations (24) and (9) yield

$$\dot{J}_2(x) = x_4\left[k_{p2}x_2 + k_{d2}f_\eta(x) + k_{d2}b_\eta(x)\tau_2\right]. \tag{25}$$

So, if the second control law is chosen to be

$$C_2:\ \tau_2 = \tilde{\tau}_2 - \lambda_2 \mathrm{sat}(x_4/\Phi_2), \tag{26}$$

where $\lambda_2 > 0$ and $\Phi_2 > 0$ are constants, and

$$\tilde{\tau}_2 := -\frac{k_{p2}x_2 + k_{d2}f_\eta(x)}{k_{d2}b_\eta(x)}, \tag{27}$$

then

$$\dot{J}_2(x) = -k_{d2}\lambda_2 b_\eta(x) x_4 \mathrm{sat}(x_4/\Phi_2) \leq 0 \tag{28}$$

holds when x is in the swing-up area because $k_{d2} > 0$, $\lambda_2 > 0$ and $b_\eta(x) > 0$. When the state x is in the swing-up area, the second link straightens out in a natural way, as shown in the next section.

Note that the second Lyapunov function contains only the angle and angular velocity of the second link. Thus, (26) only guarantees that the second link straightens out naturally. Since the energy of the acrobot may change dramatically while (26) is being applied, which is clearly undesirable, we need to improve (26) to guarantee that no abrupt change occurs, and that the energy of the acrobot increases to E_0 and is kept near that value. A fuzzy controller is employed for this purpose.

Substituting (26) into (18) yields

$$\dot{E}(x) = x_4\tilde{\tau}_2 - \lambda_2 x_4 \mathrm{sat}(x_4/\Phi_2). \tag{29}$$

In general, $\dot{E}(x)$ is not always zero, i.e., the energy changes when a control action is performed. So, we decompose λ_2 into

$$\lambda_2 = \lambda_\alpha(1 + r),\ \ -1 < r < 1, \tag{30}$$

where $\lambda_\alpha > 0$ is a constant. And we design a fuzzy controller to regulate the parameter r in order to achieve the goals mentioned above. The fuzzy controller has two inputs: the difference between the energy of the acrobot and E_0,

$$e := E(x) - E_0, \tag{31}$$

and the signal $w = x_4\tilde{\tau}_2$. The fuzzy output of the fuzzy logic is r_x. The fuzzy relations between the inputs and output are listed in Table I, where NB, NM, ZR, PM and PB stand for negative big, negative medium, zero, positive medium, and positive big, respectively.

The crisp output, r, is obtained from the center-of-gravity defuzzifier r_x. The derivative of the energy, $\dot{E}(x)$, is regulated by the output of the fuzzy controller, r, so as to make the energy of the acrobot reach the prescribed value, E_0, and maintain it almost unchanged around E_0. Since the change in energy is kept to a minimum while the acrobot straightens out in a natural way, the acrobot easily enters the attractive area.

The second switching condition is

$$T_2: J_3(x) < J_2(x), \tag{32}$$

which satisfies the condition in (16). $J_3(x)$ is the third Lyapunov function, which is explained in the next section.

TABLE I

FUZZY CONTROL RULES (INPUTS: e AND w; OUTPUT: r_x).

w \ e	NB	NM	ZR	PM	PB
NB	NB	NB	NB	NM	ZR
NM	NB	NB	NM	ZR	PM
ZR	NB	NM	ZR	PM	PB
PM	NM	ZR	PM	PB	PB
PB	ZR	PM	PB	PB	PB

E. Third control law

This subsection describes the design of the third control law, which is employed in the attractive area.

Simple calculations yield the following linear approximate model of the unstable inverted equilibrium position:

$$\dot{x} = Ax + B\tau_2, \tag{33}$$

The fact that the pair (A, B) is controllable [4], as is easily verified, enables us to design an efficient control law in the attractive area.

The third Lyapunov function, $J_3(x)$, is defined to be

$$J_3(x) = x^T P x, \tag{34}$$

where P is positive symmetric matrix.

An LQR is designed based on the linear state-space model (33) and the Lyapunov function (34) by optimizing J_a given as follows,

$$J_a = \int_0^\infty \left(x^T Q x + R\tau_2^2\right) dt, \tag{35}$$

where $Q \geq 0$ and $R > 0$. The resulting optimal control law is

$$C_3: \ \tau_2 = -Fx, \tag{36}$$

where

$$F = R^{-1} B^T P, \tag{37}$$

and $P = P^T > 0$ is a solution of the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{38}$$

(36) guarantees $\dot{J}_3(x) < 0$ for any $x \neq 0$ in the attractive area. Thus, it guarantees the convergence of the motion of the acrobot to the unstable inverted equilibrium position.

III. STABILITY FOR MULTIPLE CONTROLLERS

In this section, we examine the stability of an acrobot in the whole motion space when its motion is governed by a control law based on a non-smooth Lyapunov function.

A. Analysis of stability in swing-up area

In order to analyze the stability, the concept of an invariant set is first introduced.

Definition 1: [9] Consider the following system:

$$\dot{x} = f(x). \tag{39}$$

A set $\mathcal{M}$ is called an invariant set of (39) if any solution, $x(t)$, that belongs to $\mathcal{M}$ at the instant t_0 must belong to $\mathcal{M}$ at all past and future times:

$$x(t_0) \in \mathcal{M} \ \Rightarrow \ x(t) \in \mathcal{M}, \ \forall t \in \Re. \tag{40}$$

In addition, a set $\mathcal{M_P}$ is positively invariant if it is true at all future times only:

$$x(t_0) \in \mathcal{M_P} \ \Rightarrow \ x(t) \in \mathcal{M_P}, \ \forall t \geq t_0. \tag{41}$$

LaSalle's Invariance Theorem is used to examine the stability.

Lemma 1: [9] (LaSalle's Invariance Theorem) Let $\mathcal{M_P}$ be a positively invariant set of (39); $W : \mathcal{M_P} \to \Re_+$ be a continuously differentiable function, $W(x)$, such that $\dot{W}(x) \leq 0, \ \forall x \in \mathcal{M_P}$; $\Psi = \left\{x \in \mathcal{M_P} | \dot{W}(x) = 0\right\}$; and Ω be the largest invariant set contained in Ψ. Then, every bounded solution, $x(t)$, starting in $\mathcal{M_P}$ converges to Ω at $t \to \infty$.

It is easy to show that, a set, Ω_1, satisfying $(x_2, \ x_4) = (0, 0)$ and $E(x) = E_0$ is given by

$$\Omega_1: \ 0.5 m_{110} x_3^2 + [m_1 L_{g1} + m_2(L_1 + L_{g2})] g \cos x_1 = E_0, \tag{42}$$

where

$$\begin{aligned} m_{110} &= m_{11}(x)|_{(x_2=0, x_4=0)} \\ &= m_1 L_{g1}^2 + I_1 + m_2 L_{g2}^2 + I_2 + 2m_2 L_1 L_{g2} + m_2 L_1^2; \end{aligned} \tag{43}$$

Ω_1 is a positively invariant set; and the conditions (42) describe a periodic circular movement in x_1 and x_3.

Since the first control law, which is based on the first Lyapunov function (17), is operative before the singularity occurs, we do not need to consider the singularity while it is being employed. Thus, we have the following lemma for the first control law.

Lemma 2: The control law (20), which is based on the first Lyapunov function (17) and is employed before the first switch, makes the acrobot move towards the positively invariant set Ω_1 given by (42).

Proof: In order to apply Lemma 1, we first have to define several positively invariant sets. Consider the system (12) with the first control law (20). Clearly, $\Re^4$ is a positively invariant set of the control system. So, we have $\mathcal{M_P} = \Re^4$. Now, let us construct $\Psi = \left\{x \in \mathcal{M_P} | \dot{J}_1(x) = 0\right\}$. Note that $\dot{J}_1(x) = 0$ implies $x_4 = 0$. From (17) and (18), it is clear that these conditions mean that $J_1(x)$, x_2, and $E(x)$ are constant. There are two cases when $E(x)$ is constant: $E(x) = E_0$ and $E(x) \neq E_0$. When $E(x) = E_0$, (9) and (20) yield $x_2 = 0$. A simple calculation shows that the trajectory of the first link follows a circular orbit given by (42).

On the other hand, the case $E(x) \neq E_0$ is never true, as the following proof demonstrates.

Assume that $E(x) \neq E_0$ holds and define $E_\Delta = E(x) - E_0$. Since $x_4 = 0$ and $x_2 = x_{20} = ct.$, (9) and (20) give

$$\frac{f_{\eta 0}}{b_{\eta 0}} = \frac{k_{p1} x_{20} + k_{d1} f_{\eta 0}}{k_{d1} b_{\eta 0} + k_{e1} E_\Delta}, \tag{44}$$

where
$$f_{\eta 0} = f_\eta(x)|_{(x_2=x_{20},x_4=0)},$$
$$b_{\eta 0} = b_\eta(x)|_{(x_2=x_{20},x_4=0)} = ct.$$

This yields
$$\tau_2 = -\frac{f_{\eta 0}}{b_{\eta 0}} = -\frac{k_{p1}x_{20}}{k_{e1}E_\Delta} = ct., \tag{45}$$

Substituting (10) into the above equation, we obtain
$$f_{\eta 0} = S_1 x_3^2 + U_1 \sin x_1 + V_1 \cos x_1 = W_1. \tag{46}$$

At the same time, $E(x) = ct.$ and $x_4 = 0$ give
$$E(x) = S_2 x_3^2 + U_2 \sin x_1 + V_2 \cos x_1 = W_2. \tag{47}$$

Equations (46) and (47) yield
$$U_3 \sin x_1 + V_3 \cos x_1 = W_3. \tag{48}$$

In (46)-(48), S_i, U_j, V_j, and W_j ($i = 1, 2;\ j = 1, 2, 3$) are nonzero constants. From (48), the variable x_1 is constant. So, $x_3 = 0$. For $x_3 = 0$ and $x_4 = 0$, (8) and (9) can be written as
$$f_\mu(x) + b_\mu(x)\tau_2 = 0, \tag{49}$$
$$f_\eta(x) + b_\eta(x)\tau_2 = 0. \tag{50}$$

(10), (11), (49) and (50) yield
$$\det\{M(x)\}|_{(x_3=0,x_4=0)} = 0, \tag{51}$$

which contradicts the fact that the determinant of the inertia matrix, $M(x)$, is positive. Therefore, only the case $E(x) = E_0$ is valid in Ψ.

Since Ω_1, which is given by (42), is a positively invariant set and since $\Psi = \Omega_1$, Ω_1 is the largest invariant set contained in Ψ. According to Lemma 1, the control law (20), which is based on the first Lyapunov function (17) and is operative only before the first switch, makes the acrobot move toward the invariant set Ω_1. ■

The stability of the control law (26) is guaranteed by the following lemma.

Lemma 3: The control law (26), which is based on the second Lyapunov function (24), makes the states of the second link of the acrobot converge to the positively invariant set $\Omega_2 = \{x \in \Re^4 | x_2 = 0, x_4 = 0\}$.

The proof is based on Lemma 1 and is similar to that for Lemma 2. Thus, it is omitted.

B. Stability for multiple controllers

A system, e.g., (12), is stabilizable if there exists a candidate Lyapunov function, $J(x)$, for which decreases monotonically in the whole motion space. First, the following assumptions and definitions are given.

Assumption 1: There exist Lyapunov functions, $J_1(x)$, $J_2(x)$ and $J_3(x)$ for each pair (C_1, Σ_{s1}), (C_2, Σ_{s2}) and (C_3, Σ_a), respectively.

Assumption 2: Σ_{s1}, Σ_{s2} and Σ_a together cover the whole motion space Σ, i.e.,
$$\Sigma = \Sigma_{s1} \bigcup \Sigma_{s2} \bigcup \Sigma_a. \tag{52}$$

Definition 2: [10] (Non-smooth Lyapunov Function) A non-smooth Lyapunov function, $J(x)$, for the system (12) in the whole motion space, Σ, is a Lyapunov function given by
$$J(x) = \bigcup_{i=1}^{3} J_i(x), \tag{53}$$

which satisfies
$$J(x(t_2)) < J(x(t_1)),\ t_2 > t_1,\ 0 \le t_1, t_2 < \infty. \tag{54}$$

Definition 3: [10] (Minimum-switching strategy) Control law C_i is employed when $J_i(x) < J_{i-1}(x)$; otherwise control law C_{i-1} $(i = 2, 3)$ is employed.

Note that in Definition 2, the condition $\dot{J}(x) < 0$ for an ordinary differentiable function is replaced by a monotonically decreasing condition for an undifferentiable non-smooth function. The stability of our control method is guaranteed by the following theorem.

Theorem 1: For the non-smooth Lyapunov function $J(x)$ (53) employed for the system (12), the closed-loop system is stable under the assumptions (1)-(2) if the minimum-switching strategy is used.

Proof: The first Lyapunov function is chosen to be (17). It increases the energy and stretches out the second link. The second Lyapunov function is chosen to be (24). It avoids the singularity produced by the first control law, and continues to stretch out the second link. A suitable choice of the coefficients in (17) and (24) guarantees that the second Lyapunov function is less than the first one.

The constant Δ_1 in (17) and (24) is used to satisfy the condition (54) in Definition 2, i.e., to guarantee that $J_3(x) < J_2(x)$. It is selected to be
$$\Delta_1 = J_3(x)|_{x_\alpha}, \tag{55}$$

where $x_\alpha \in \Sigma_a$ is the state for which the switching condition (16) is satisfied for the first time and control is first switched from the second to the third law. Note that, when (23) (or (16)) is satisfied and the controller is switched from C_1 to C_2 (or from C_2 to C_3), $J_2(x) < J_1(x)$ (or $J_3(x) < J_2(x)$) also holds.

The sets for the switching condition $\{T_1,\ T_2\}$ specify the timing of the switch between controllers. The control law governing the whole motion space is related to the non-smooth Lyapunov function, $J(x)$. Note that $J(x)$ is decreasing at the switching points; so, the minimum-switching strategy guarantees that no switching back and forth between two controllers occurs. This guarantees the stability of the system in the whole motion space. ■

IV. Simulations

This section presents simulation results that demonstrate the validity of the control strategy explained above.

The parameters of the acrobot are $m_1 = 1$ kg, $m_2 = 1$ kg, $I_1 = 0.083$ kg $\cdot$ m^2, $I_2 = 0.33$ kg $\cdot$ m^2, $L_1 = 1$ m, $L_2 = 2$ m, $L_{g1} = 0.5$ m and $L_{g2} = 1$ m.

The parameters in (16), (17), (20), (23), (24), (26), and (30) are selected to be $\beta_1 = \beta_2 = \frac{\pi}{6}$, $\epsilon = 1.2$ J, $k_{p1} = k_{d1} = 1$,

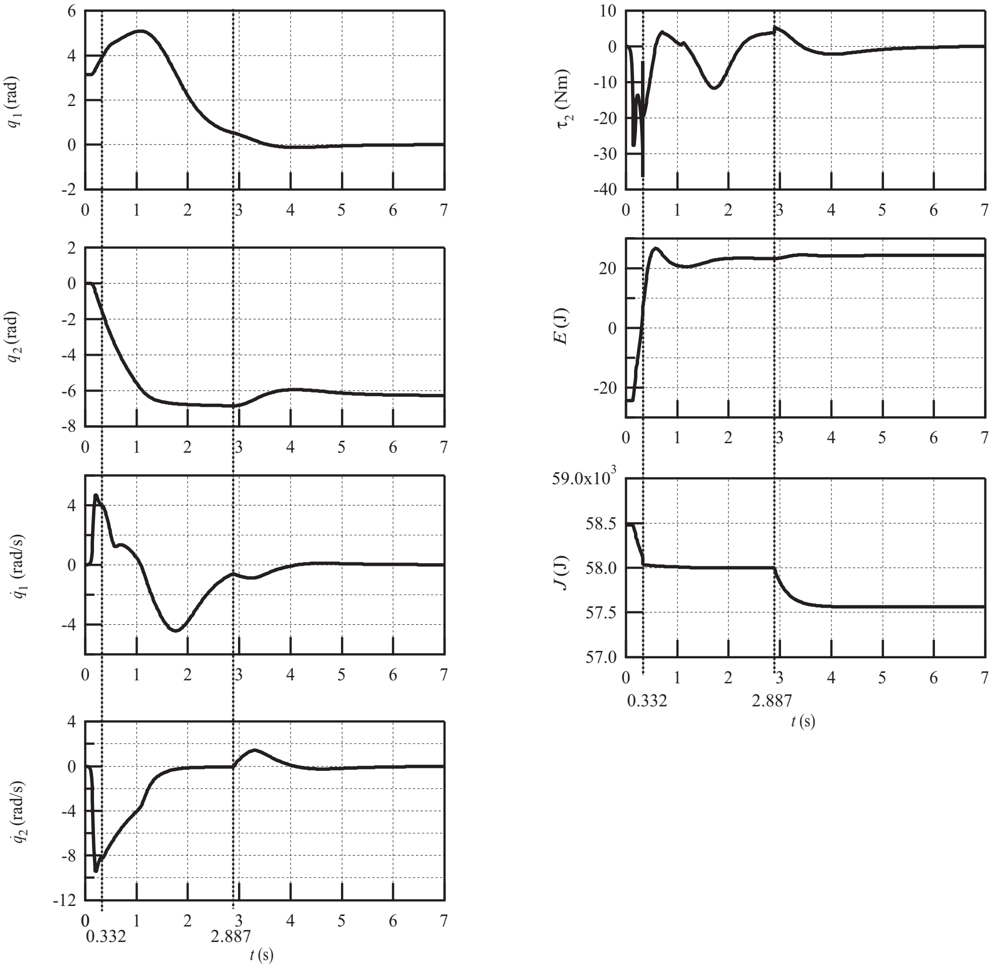

Fig. 2. Simulation results for switching of control laws based on a non-smooth Lyapunov function

$k_{e1} = 0.2$, $\lambda_1 = 38$, $\Phi_1 = 10$, $\zeta = -2$, $k_{p2} = k_{d2} = 1$, $\Phi_2 = 5$ and $\lambda_\alpha = 0.5$.

Figure 2 shows simulation results for the initial state $x = [\pi \ 0 \ 0 \ 0]^T$. The non-smooth Lyapunov function decreased monotonically. The control law successfully regulated the motion of the acrobot, making it smoothly converge to the unstable inverted equilibrium position. These results demonstrate that control based on a non-smooth Lyapunov function is very effective.

V. Conclusions

This paper has provided the control strategy to guarantees the stability of the acrobot in the whole motion space under a minimum-switching strategy based on non-smooth Lyapunov function consisting of a combination of three Lyapunov functions. This paper has also clearly shown how to choose three Lyapunov function to obtain the control objective. The validity of the strategy has been demonstrated through simulations. The results showed that the acrobot swings up and is balanced at the unstable inverted equilibrium position very quickly.

References

[1] M. W. Spong, "The swing up control problem for the acrobot", *IEEE Contr. Syst. Mag.*, vol. 15, pp. 49-55, 1995.

[2] S. Brown and K. Passin, "Intelligent control for acrobot", *J. intelligent and robotic systems*, vol. 18, pp. 209-248, 1997.

[3] J.-H. She, Y. Ohyama, K. Hirano, and M. Inoue, "Motion control of acrobot using time-state control form", *Proc. IASTED Int. Conf. Contr. Applications*, pp. 171-174, 1998.

[4] X.-Z., Lai, J.-H. She, Y. Ohyama, and Z. Cai, "A fuzzy control strategy for acrobot combining model-free and model-based control", *IEE Proc. Contr. Theory and Appl.*, vol. 146, pp. 505-511, 1999.

[5] K. J. Åström and K. Furuta "Swinging up a pendulum by energy control", *Automatica*, vol. 36, pp. 287-295, 2000.

[6] I. Fantoni, R. Lozano, and M. W. Spong, "Energy based control of the pendubot", *IEEE Trans. on Automat. Contr.*,vol. 45, pp. 725-729, 2000.

[7] X. Xin and M. Kaneda, "New analytical results of the energy based swinging up control of the acrobot", *Proc. 43rd IEEE Conf. Decision Contr.*, pp. 704-709, 2004.

[8] X.-Z. Lai, M. Wu, and J.-H. She, "Control of acrobot based on Lyapunov function", *J. Central South University of Technology*, vol. 11, pp. 210-215, 2004.

[9] M. Krstić, I. Kanellakopoulos and P. Kokotović, *Nonlinear and Adaptive Control Design*, John Willey & Sons, Inc., New York, NY, 1995.

[10] J. Malmborg, B. Bernhardsson, and K. J. Åström, "A stabilizing switching scheme for multi controller systems," *IFAC 13th World Cong.*, pp. 229-234, 1996.