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MOTION CONTROL OF ACROBOT USING TIME-STATE CONTROL FORM

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ABSTRACT

This paper proposes a design method for the control of an acrobot. The concept of *time state* is employed in the design. First, the motion space is divided into two subspaces: one around an equilibrium point (SI) and one for the remainder (SII). Next, a nonlinear control law is designed for SII by introducing a virtual time axis, and a quadratic optimal linear control law is designed for SI. Finally, a combination of these two control laws enables the acrobot to be driven from any initial position to the equilibrium point and stabilized at that point.

Key words: acrobot, underactuated mechanical system, time state, exact linearization, Lyapunov function, linear optimal control.

1. INTRODUCTION

Underactuated mechanical systems are subject to nonholonomic constraints ([1]) that make such systems very hard to control ([2]). Over the last decade, this control has been the subject of controversy, and a considerable number of studies have been done on it (e.g. [3], [4], [5] and [6]). On the other hand, some control problems become very simple if they are formulated in terms of a new axis other than time ([7], [8] and [9]).

An acrobot is a two-degree-of-freedom planar robot with a single actuator, as shown in Fig. 1. The first joint is passive and the second one is actuated. Through movement of the second joint, it operates in a vertical plane. Due to gravity, an acrobot has a very important characteristic that general underactuated mechanical systems do not possess, namely, it *can* be stabilized by means of smooth state feedback. The control objective is to drive the acrobot from an initial position around the stable position $q_0 = [q_{10} \quad q_{20}]^T = [\pi \quad 0]^T$ to the unstable inverted final position $q_T = [q_{1T} \quad q_{2T}]^T = [0 \quad 0]^T$, and balance it around q_T. The problem of controlling an acrobot has been investigated by many researchers, and many methods, for example, nonlinear approximation ([10]), partial feedback linearization ([5]) etc., have been presented.

This paper proposes a method of controlling an acrobot based on the concept of *time state*. First, the motion space is divided into two subspaces: one around the unstable inverted position (SI) and one for the remainder (SII). Next, real time is converted to virtual time, and the dynamics of the acrobot are exactly linearized in the virtual time domain. Moreover, a nonlinear control law that guarantees the stability of the system in SII is designed in the virtualtime

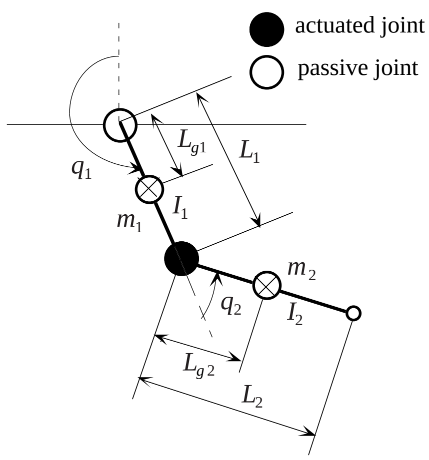

Fig. 1. Model of acrobot.

domain. Then, a linear control law that guarantees the stability of the system in SI is designed in the time domain. Finally, a combination of these two control laws enables the acrobot to be driven from any initial position to the final position q_T.

2. MODELS OF ACROBOT

Consider the acrobot shown in Fig. 1. Its dynamic equations are

$$m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q,\dot{q}) + g_1(q) = 0\,, \qquad (1)$$

$$m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q,\dot{q}) + g_2(q) = \tau\,, \qquad (2)$$

where

$$\begin{cases} q := [q_1 \quad q_2]^T \\ m_{11}(q) = I_1 + I_2 + 2m_2L_1L_{g2}\mathrm{C}q_2 + m_2L_1^2 \\ m_{22}(q) = I_2 \\ m_{12}(q) = m_{21}(q) = I_2 + m_2L_1L_{g2}\mathrm{C}q_2 \\ h_1(q,\dot{q}) = -m_2L_1L_{g2}\dot{q}_2(2\dot{q}_1 + \dot{q}_2)\mathrm{S}q_2 \\ h_2(q,\dot{q}) = m_2L_1L_{g2}\dot{q}_1^2\mathrm{S}q_2 \\ g_1(q) = -(m_1L_{g1} + m_2L_1)g\mathrm{S}q_1 - m_2L_{g2}g\mathrm{S}(q_1+q_2) \\ g_2(q) = -m_2L_{g2}g\mathrm{S}(q_1+q_2), \end{cases} \qquad (3)$$

τ is the applied torque, m_i and I_i are the mass and rotational inertia of link i $(i = 1,2)$, respectively, and

$$\cos q := \mathrm{C}q \quad \sin q := \mathrm{S}q$$

for simplicity.

In particular, if we define

$$x = [q_1 \quad q_2 \quad \dot{q}_1 \quad \dot{q}_2]^T,$$

then we can use the following approximation

$$\begin{cases} \mathrm{C}q_i = 1 & \mathrm{S}q_i = q_i \\ \mathrm{S}(q_i + q_j) = q_i + q_j & \dot{q}_i\dot{q}_j = 0 \end{cases} \quad (i, j = 1,2)$$

around the equilibrium point $x_T = 0$. Some simple calculations yield the following linear model.

$$\dot{x} = Ax + B\tau, \qquad (4)$$

$$\begin{cases} A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-m_{22}\bar{g}_1 + m_{12}\bar{g}_2}{\Delta} & \dfrac{m_{12} - m_{22}}{\Delta}\bar{g}_2 & 0 & 0 \\ \dfrac{-m_{21}\bar{g}_1 + m_{11}\bar{g}_2}{\Delta} & \dfrac{m_{21} - m_{11}}{\Delta}\bar{g}_2 & 0 & 0 \end{bmatrix} \\ B = \begin{bmatrix} 0 & 0 & -\dfrac{m_{12}}{\Delta} & \dfrac{m_{11}}{\Delta} \end{bmatrix}^T, \end{cases} \qquad (5)$$

where,

$$\begin{cases} \Delta = m_{11}m_{22} - m_{21}m_{12} \\ m_{11} = I_1 + I_2 + 2m_2L_1L_{g2} + m_2L_1^2 \\ m_{22} = I_2 \\ m_{12} = m_{21} = I_2 + m_2L_1L_{g2} \\ \bar{g}_1 = -(m_1L_{g1} + m_2L_1 + m_2L_{g2})g \\ \bar{g}_2 = -m_2L_{g2}g. \end{cases}$$

The following characteristics of the system can be easily verified ([1], [11] and [12]):

1) The acrobot is second-order nonholonomic.
2) The acrobot is stabilizable by smooth state feedback.
3) The acrobot cannot be exactly linearized in the time domain.

To simplify the control problem, we choose a small positive number ε and use it to divide the motion space into two subspaces. If we define

$$X = \big[|q_1| \quad |q_2| \quad |\dot{q}_1| \quad |\dot{q}_2|\big]^T$$

and

$$\lambda = [\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4]^T \quad (\lambda_i \geq 0,\ \sum_{i=1}^{4}\lambda_i > 0),$$

then the subspaces are defined as

SI: $\lambda^T X \leq \varepsilon$,

SII: $\lambda^T X > \varepsilon$.

It is clear that the system in SI can be treated as a linear system and we can use the linear model (4) to design a stabilizing control law easily. So, the main problem in control is how to drive the system from SII to SI.

3. DESIGN OF CONTROL LAWS

3.1 Control law in the subspace SII

3.1.1 Control law in the virtual time domain

We introduce the following time state

$$\frac{d\xi}{dt} = \mu(q,t) \qquad (6)$$

to change real time to a new virtual time, ξ. In view of (6), the relationship between the real and virtual time domains can be summarized as $(i = 1,2)$

$$\begin{cases} \xi = \int_0^t \mu(q,t)dt \\ \dot{q}_i = \dfrac{dq_i}{d\zeta}\dfrac{d\zeta}{dt} = \mu q_i' \quad \ddot{q}_i = \dfrac{d(\mu q_i')}{d\zeta}\dfrac{d\zeta}{dt} = \mu'\mu q_i' + \mu^2 q_i''. \end{cases} \qquad (7)$$

Accordingly, the dynamics of the acrobot in the virtual time domain are

$$m_{11}(q)q_1'' + m_{12}(q)q_2'' + \hat{h}_1(q,q') + \hat{g}_1(q) = 0\,, \qquad (8)$$

$$m_{21}(q)q_1'' + m_{22}(q)q_2'' + \hat{h}_2(q,q') + \hat{g}_2(q) = \tau/\mu^2, \qquad (9)$$

where

$$\begin{cases} \hat{h}_1(q,q') = \dfrac{h_1(q,\mu q')}{\mu^2} + \dfrac{\mu'}{\mu}\{m_{11}(q)q_1' + m_{12}(q)q_2'\} \\ \hat{h}_2(q,q') = \dfrac{h_2(q,\mu q')}{\mu^2} + \dfrac{\mu'}{\mu}\{m_{21}(q)q_1' + m_{22}(q)q_2'\} \\ \hat{g}_1(q) = \dfrac{g_1(q)}{\mu^2} \\ \hat{g}_2(q) = \dfrac{g_2(q)}{\mu^2}. \end{cases}$$

Thus, if we define two new control inputs as

$$u_1 = -\frac{m_{12}(q)q_2'' + \hat{h}_1(q,q') + \hat{g}_1(q)}{m_{11}(q)}, \qquad (10)$$

$$u_2 = \frac{\tau / \mu^2 - m_{21}(q)q_1'' - \hat{h}_2(q,q') - \hat{g}_2(q)}{m_{22}(q)}, \tag{11}$$

then the system becomes

$$q_1'' = u_1, \tag{12}$$

$$q_2'' = u_2. \tag{13}$$

Clearly, the dynamics of the acrobot are exactly linearized in the virtual time domain.

If a Lyapunov function is defined as

$$V(x) = \frac{1}{2}x^T x = \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 + \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2, \tag{14}$$

then

$$\dot{V}(x) = \frac{dV(x)}{dt} = q_1\dot{q}_1 + q_2\dot{q}_2 + \dot{q}_1\ddot{q}_1 + \dot{q}_2\ddot{q}_2$$
$$= \mu q_1 q_1' + \mu q_2 q_2' + \mu q_1'(\mu'\mu q_1' + \mu^2 q_1'')$$
$$+ \mu q_2'(\mu'\mu q_2' + \mu^2 q_2'').$$

It is clear that the following control law

$$\begin{cases} u_1 = -\dfrac{q_1}{\mu^2} - \dfrac{\mu'}{\mu}q_1' - k_{11}\mu q_1'\left(\sqrt{q_1'^4 + \dfrac{1}{\mu^4}} - q_1'^2\right) - k_{12}q_2' \\ u_2 = -\dfrac{q_2}{\mu^2} - \dfrac{\mu'}{\mu}q_2' - k_{22}\mu q_2'\left(\sqrt{q_2'^4 + \dfrac{1}{\mu^4}} - q_2'^2\right) + k_{12}q_1' \end{cases} \tag{15}$$

makes $\dot{V}(x) \le 0$ if k_{11} and k_{22} are chosen such that

$$k_{11} > 0, \quad k_{22} > 0. \tag{16}$$

3.1.2 Control law in the time domain

(10) and (15) yield

$$k_{12}(m_{12}\dot{q}_1 - m_{11}\dot{q}_2)\mu - k_{11}m_{11}\dot{q}_1(\sqrt{\dot{q}_1^2 + 1} - \dot{q}_1^2)$$
$$-k_{22}m_{22}\dot{q}_2(\sqrt{\dot{q}_2^4 + 1} - \dot{q}_2^2) \tag{17}$$
$$-(m_{11}q_1 + m_{12}q_2) + h_1 + g_1 = 0.$$

So, the transfer rate μ between the virtual time ζ and the real time t can be obtained by solving the above equation:

$$\mu = \frac{1}{k_{12}(m_{12}\dot{q}_1 - m_{11}\dot{q}_2)}\{k_{11}m_{11}\dot{q}_1(\sqrt{\dot{q}_1^2 + 1} - \dot{q}_1^2)$$
$$+k_{22}m_{22}\dot{q}_2(\sqrt{\dot{q}_2^4 + 1} - \dot{q}_2^2) + (m_{11}q_1 + m_{12}q_2) \tag{18}$$
$$-h_1 - g_1\}.$$

Remark: We cannot obtain a μ when $m_{12}\dot{q}_1 - m_{11}\dot{q}_2 = 0$. But this problem can be solved by choosing another control law that has singularities different from $m_{12}\dot{q}_1 - m_{11}\dot{q}_2 = 0$ and switching control laws from one to the other when the acrobot approaches to a singularity.

Implementing the control law (15) in the real time domain yields

$$\begin{cases} u_1 = -\dfrac{q_1}{\mu^2} - \dfrac{\dot{\mu}}{\mu^3}\dot{q}_1 - \dfrac{k_{11}}{\mu^2}\dot{q}_1(\sqrt{\dot{q}_1^4 + 1} - \dot{q}_1^2) - \dfrac{k_{12}}{\mu}\dot{q}_2 \\ u_2 = -\dfrac{q_2}{\mu^2} - \dfrac{\dot{\mu}}{\mu^3}\dot{q}_2 - \dfrac{k_{22}}{\mu^2}\dot{q}_2(\sqrt{\dot{q}_2^4 + 1} - \dot{q}_2^2) + \dfrac{k_{12}}{\mu}\dot{q}_1. \end{cases} \tag{19}$$

Therefore, the control toque in the real time domain is given by

$$\tau = \mu^2(m_{21}u_1 + m_{22}u_2) + h_2 + g_2$$
$$+\mu'\mu(m_{21}q_1' + m_{22}q_2')$$
$$= -(m_{21}q_1 + m_{22}q_2) - k_{11}m_{21}\dot{q}_1(\sqrt{\dot{q}_1^4 + 1} - \dot{q}_1^2) \tag{20}$$
$$-k_{22}m_{22}\dot{q}_2(\sqrt{\dot{q}_2^4 + 1} - \dot{q}_2^2)$$
$$-k_{12}\mu(m_{21}\dot{q}_2 - m_{22}\dot{q}_1) + h_2 + g_2.$$

It can be observed from (17) that any μ will satisfy this equation at the equilibrium point $x_T = 0$. So, the numerical conditions are unsuitable for calculating the control law in the subspace SI. For this reason, we need to design another control law in this subspace.

3.2 Control law in the subspace SI

The control law in SI is designed based on the linear model (4) by optimizing the following performance index:

$$J = \int_0^\infty (x^T Q x + R\tau^2)dt \quad (Q \ge 0, R > 0). \tag{21}$$

The resulting control law is

$$\tau = -Fx, \tag{22}$$

where

$$F = R^{-1}BP, \tag{23}$$

and P is given by the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{24}$$

4. SIMULATION

The parameters of the acrobot used in the simulation are listed in Table 1. Let's consider the problem of driving the acrobot from the initial state $x_0 = [q_{10} \quad q_{20} \quad \dot{q}_{10} \quad \dot{q}_{20}]^T = [\pi \quad 0.2 \quad 0.0 \quad 0.5]^T$ to the final state $x_T = 0$. The parameters

$$\varepsilon = 0.8, \quad \lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 0, \quad \lambda_4 = 0. \tag{25}$$

were chosen to divide the motion space. For the design of the control law in SI,

$$Q = \mathrm{diag}\{1,\ 1,\ 1,\ 1\}, \quad R = 10 \tag{26}$$

were selected, and k_{11}, k_{12} and k_{22} were chosen to be

$$k_{11} = 15, \quad k_{12} = 0.2, \quad k_{22} = 10. \tag{27}$$

The simulation results are shown in Fig. 2. It can be seen from the response that the acrobot stored energy when it swung up in the subspace SII. The control law changed from the nonlinear one to the linear one at t = 18.885 sec. Then it reached the final state around t = 20 sec.

Table 1. Parameters of acrobot.

	Link 1	Link 2
m_i [kg]	7.17	5.45
L_i [m]	0.25	0.3
L_{gi} [m]	8.25×10^{-2}	2.42×10^{-1}
I_i [kgm]	4.88×10^{-2}	3.19×10^{-1}

5. CONCLUSIONS

This paper presents a design method for the control of an acrobot. The motion space is first divided into two subspace: SI is around the unstable inverted final position and SII is the remainder. It has been shown that the dynamics of the system in SII can be exactly linearized in a virtual time domain by employing the concept of time state to introduce virtual time. A nonlinear control law has been designed in SII, and the stability of the system is guaranteed by a Lyapnov function. The validity of the proposed method has been demonstrated through simulations, though experimental verification awaits further study.

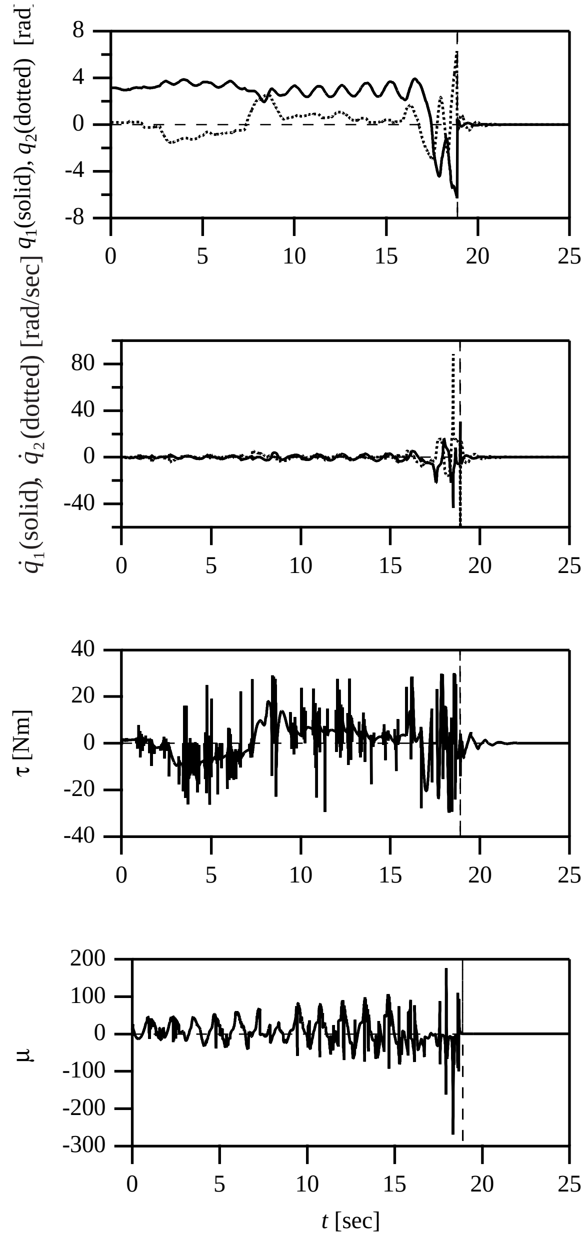

Fig. 2. Simulation results.

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