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A Design Methodology for Robust Two-Degree-of-Freedom Digital Preview Tracking Controllers

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Abstract

This paper presents a design method for digital tracking control systems in which the plant has structured uncertainties. A two-degree-of-freedom control system configuration is utilized to achieve the desired feedback and input-output performances independently. First, sampled-data $\mathcal{H}_\infty$ control and linear matrix inequality approaches are used to design a static state feedback controller and a reduced-order output feedback controller. Then, the parameterization of a feedforward controller is carried out based on the feedback controller, in which the free parameter is chosen to achieve the desired transient response using a preview strategy.

1. Introduction

Tracking control systems require not only good closed-loop performance, but also good tracking performance. However, there is usually a trade-off between them in a conventional one-degree-of-freedom (ODF) control system. So, it is difficult to design a satisfactory controller that meets both requirements. In contrast, a two-degree-of-freedom (TDF) control system processes the reference input and the plant output separately, thus enables the independent design of the closed-loop and tracking performances. More specifically, the required performances can be achieved by designing a suitable feedback and feedforward controller, respectively (e.g., Vidyasagar, 1985; Hara and Sugie, 1988).

Over the past few years, sampled-data $\mathcal{H}_\infty$ control, which handles the continuous uncertainties of a plant directly, has provoked a great deal of interest (e.g., Bamieh and Pearson, 1992; Kabamba and Hara, 1993). It motivates this study, which has the goal of designing a digital robust feedback controller in a TDF control system that handles continuous uncertain plants directly.

On the other hand, it is well known that the performance of a control system can be improved by constructively using information about future inputs (Tomizuka, 1993). Funahashi and Katoh (1992) proposed a design method for a preview step-type servo system that employs a TDF system configuration. Their method is based on parameterization of the stabilizing controllers of a TDF control system, and a preview action is introduced by expanding the parameter related to the tracking performance into an improper stable class.

This paper presents a design method for a TDF digital tracking control system for a continuous plant with uncertainties. Regarding the problem of designing the feedback controller, it is first formulated as a sampled-data $\mathcal{H}_\infty$ control problem, and is then transformed into a discrete-time $\mathcal{H}_\infty$ control problem. Since the order of an $\mathcal{H}_\infty$ controller is usually very high, the results in Xin *et al.* (1996), in which a reduced-order controller was designed based on linear matrix inequalities (LMI) (e.g., Gahinet and Apkarian, 1994; Iwasaki and Skelton, 1994), are used to obtain static state feedback and reduced-order output feedback $\mathcal{H}_\infty$ controllers. Regarding the design of the feedforward controller, first the parameterization of the controller is carried out based on the feedback controller. Then a method of designing an optimal preview tracking feedforward controller, that extends the basic idea proposed by Funahashi and Katoh (1992) to a general servo system, is explained.

Throughout this paper, z denotes a Z-transform variable, and λ denotes the delay operator such that $\lambda = z^{-1}$ holds. $\mathbf{RH}_\infty$ is a set of real-rational functions in λ which have no poles in the closed unit circle. $\mathbf{R}_*$ indicates a set of real-rational functions in λ which have no poles in the closed unit circle except for the origin. $\mathbf{R}[\lambda]$ and $\mathbf{R}[z]$ are rings of polynomials in λ ($\subset \mathbf{RH}_\infty$) and in z ($\subset \mathbf{R}_*$), respectively. $\Omega[a(\lambda)]$ is the set of zeros of the polynomial $a(\lambda)$; and $a^+(\lambda)$ and $a^-(\lambda)$, which satisfy $a(\lambda) = a^+(\lambda)a^-(\lambda)$, denote two polynomials with roots in and outside the closed unit circle, respectively. G indicates a continuous-time or discrete-time system, while $\mathcal{G}$ indicates a hybrid system that contains both continuous and discrete-time time-invariant sub-systems.

All of the proofs are omitted for brevity.

2. Problem Formulation

Consider the TDF tracking control system configuration shown in Fig. 1. $P(s)$ is a plant with structured uncertainties:

$$\begin{cases} \dot{x}_P(t) = (A_P + \Phi\Gamma(t)\Psi_A)x_P(t) + (B_P + \Phi\Gamma(t)\Psi_B)u_P(t) \\ y(t) = C_P x_P(t) \\ y_F(t) = C_F x_P(t) \\ \Gamma^T(t)\Gamma(t) \le I, \end{cases} \tag{1}$$

where $x_P(t) \in \mathbf{R}^{n_P}$, $y(t) \in \mathbf{R}$, $u_P(t) \in \mathbf{R}$ and $y_F(t) \in \mathbf{R}^{m_P}$ are the state, observed output, control input and measured variable of the plant, respectively. In particular, $C_F = I_{n_P}$ means the state feedback, and $C_F = C_P$ means the output feedback. Without loss of generality, $C_P = [\ c_{P1} \quad 0\ ]$, $c_{P1} \neq 0$ ($c_{P1} \in \mathbf{R}$) is assumed. Let the reference input be

$$\begin{cases} r(\lambda) := \dfrac{\bar{r}(\lambda)}{\phi_R(\lambda)} \\ \bar{r}(\lambda) := r_0 + r_1\lambda + \ldots + r_{L-1}\lambda^{L-1} \\ \phi_R(\lambda) := 1 + \phi_1\lambda + \ldots + \phi_L\lambda^L, \end{cases} \tag{2}$$

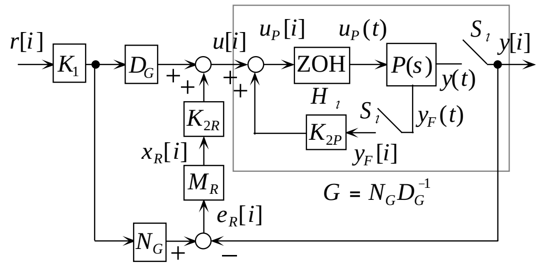

Figure 1: Configuration of two-degree-of-freedom robust tracking control system.

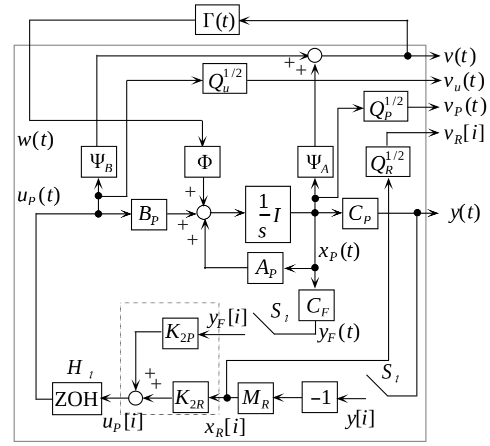

Figure 2: Design of feedback controller.

with all roots of $\phi_R(\lambda) = 0$ being in the closed unit circle. Then the state space representation of the internal model of the reference input, $M_R(\lambda)$, is

$$\begin{cases} x_R[i+1] = A_R x_R[i] + B_R e_R[i] \\ A_R = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -\phi_L & -\phi_{L-1} & \dots & -\phi_1 \end{bmatrix} \in \mathbf{R}^{L\times L} \\ B_R = [0 \quad \dots \quad 0 \quad 1]^T \in \mathbf{R}^{L\times 1}. \end{cases} \tag{3}$$

Now, letting the pulse transfer function of the nominal plant $P_0(\lambda)$ ($\Gamma(t) = 0$) with local feedback controller $K_{2P}(\lambda)$ be $G = N_G D_G^{-1}$ ($N_G, D_G \in \mathbf{R}[\lambda]$) and applying the TDF control system configuration proposed by Hara and Sugie (1988) to G yield the configuration of the TDF robust tracking control system (Fig. 1). In this figure, $K_1(\lambda)$ is the feedforward controller, and is chosen to be any stable pulse transfer function.

In Fig. 1, the feedback controller $K_2 = [\ K_{2P} \quad K_{2R}\ ]$ is defined in terms of $y_F[i]$ and $x_R[i]$ of M_R to be

$$u_P[i] = K_2 \begin{bmatrix} y_F[i] \\ x_R[i] \end{bmatrix}. \tag{4}$$

This paper considers the following design problem for robust tracking control systems.

(a) Design a reduced-order feedback controller K_2 in Eq. (4) with an order less than n_P, that robustly stabilizes the control system in Fig. 1.

(b) Design a feedforward controller that yields the desired nominal input-output tracking performance.

The following assumptions are necessary for the solvability of the problem.

(A1) $P(s)$ in Eq. (1) is stabilizable and detectable.

(A2) The sampling period, τ, is chosen so that the discrete plant obtained by putting a zero-order holder, $\mathcal{H}_\tau$, and a sampler, $\mathcal{S}_\tau$, at the input and output of $P(s)$, respectively, is stabilizable and detectable.

(A3) $\phi_R(\lambda)$ has no zeros in common with the pulse transfer function of the nominal plant $P_0(\lambda)$.

3. Design of Feedback Controller

Redrawing Fig. 1 with $r[i] = 0$ gives Fig. 2, in which the two new signals $v(t)$ and $w(t)$ are defined to be the input and output of the uncertainty $\Gamma(t)$, respectively; and the other new signals, $v_u(t)$, $v_P(t)$ and $v_R[i]$, are the control input, and the states of the plant and internal model weighted by positive semi-definite matrices $Q_u^{1/2}$, $Q_P^{1/2}$ and $Q_R^{1/2}$, respectively.

The condition for the robust stability of the control system in Fig. 1 is as follows (Sivashankar and Khargonekar, 1993).

LEMMA 1 *The control system in Fig. 1 is robustly stable if the following holds in Fig. 2:*

$$\|\mathcal{G}_{\mathcal{P}}\|_\infty := \sup_{w(t)\in\mathbf{L}_2} \frac{\|v(t)\|_2}{\|w(t)\|_2} < 1. \tag{5}$$

Let

$$v_a := [\ v(t) \quad v_u(t) \quad v_P(t) \quad v_R[i]\ ]^T,$$

then the design problem for the feedback controller is formulated as:

Find a reduced-order feedback controller $K_2(\lambda)$ in Eq. (4) that internally stabilizes the generalized plant $\mathcal{P}_\mathcal{S}$ described by

$$\begin{bmatrix} v_a \\ \hdashline y_F[i] \\ x_R[i] \end{bmatrix} = \mathcal{P}_\mathcal{S} \begin{bmatrix} w(t) \\ \hdashline u_P[i] \end{bmatrix} \tag{6}$$

and satisfies $\|\mathcal{G}_{\mathcal{P}_\mathcal{S}}\|_\infty < 1$, where $\mathcal{G}_{\mathcal{P}\mathcal{S}} = \mathcal{P}_\mathcal{S} * K_2 = \mathcal{P}_{\mathcal{S}11} + \mathcal{P}_{\mathcal{S}12} K_2 (I - \mathcal{P}_{\mathcal{S}22} K_2)^{-1} \mathcal{P}_{\mathcal{S}21}$, and $\mathcal{P}_\mathcal{S}$ is given by

$$\mathcal{P}_\mathcal{S} = \begin{bmatrix} \mathcal{P}_{\mathcal{S}11} & \mathcal{P}_{\mathcal{S}12} \\ \mathcal{P}_{\mathcal{S}21} & \mathcal{P}_{\mathcal{S}22} \end{bmatrix} = \left[\begin{array}{cc|c:c} A_R & -B_R \mathcal{S}_\tau C_P & 0 & 0 \\ 0 & A_P & \Phi & B_P \mathcal{H}_\tau \\ \hline 0 & \Psi_A & 0 & \Psi_B \mathcal{H}_\tau \\ 0 & 0 & 0 & Q_u^{1/2} \mathcal{H}_\tau \\ 0 & Q_P^{1/2} & 0 & 0 \\ Q_R^{1/2} & 0 & 0 & 0 \\ \hdashline 0 & \mathcal{S}_\tau C_F & 0 & 0 \\ I_L & 0 & 0 & 0 \end{array}\right].$$

This design problem for the feedback controller can easily be converted to an equivalent discrete-time $\mathcal{H}_\infty$ control problem (She and Nakano, 1996).

Let the equivalent generalized plant be

$$P_e(\lambda) := \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & 0 & 0 \end{array}\right], \tag{7}$$

then the design problem can be solved by the following lemma (Gahinet and Apkarian, 1994).

LEMMA 2 *The $\mathcal{H}_\infty$ control problem for the discrete-time system (7) is solvable if and only if $\mathcal{L}_D \neq \emptyset$ where*

$$\mathcal{L}_D := \left\{(X,Y) : X \in \mathcal{L}_B, Y \in \mathcal{L}_C, \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0\right\}, \tag{8}$$

$$\mathcal{L}_B := \left\{X : X = X^T > 0, \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^\perp M_B \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^{\perp T} < 0\right\}, \tag{9}$$

$$\mathcal{L}_C := \left\{Y : Y = Y^T > 0, \begin{bmatrix} C_2^T \\ 0 \end{bmatrix}^\perp M_C \begin{bmatrix} C_2^T \\ 0 \end{bmatrix}^{\perp T} < 0\right\}, \tag{10}$$

$$M_B := \begin{bmatrix} AXA^T - X + B_1B_1^T & AXC_1^T + B_1D_{11}^T \\ C_1XA^T + D_{11}B_1^T & C_1XC_1^T + D_{11}D_{11}^T - I \end{bmatrix},$$

$$M_C := \begin{bmatrix} A^TYA - Y + C_1^TC_1 & A^TYB_1 + C_1^TD_{11} \\ B_1^TYA + D_{11}^TC_1 & B_1^TYB_1 + D_{11}^TD_{11} - I \end{bmatrix}.$$

Suppose $\mathcal{L}_D \neq \emptyset$. Then there exists an $\mathcal{H}_\infty$ controller of order n_d satisfying:

$$n_d \leq \operatorname{rank}(Y - X^{-1}). \tag{11}$$

If Lemma 2 were directly used to find an $\mathcal{H}_\infty$ controller, the order of the feedback controller would generally be $n_L = L + n_P$. To design reduced-order feedback controllers, we have the following results.

THEOREM 1 *Suppose the discrete-time $\mathcal{H}_\infty$ control problem for the generalized plant (7) with $C_F = C_P$ is solvable. Let the LMI solution be $(X,Y) \in \mathcal{L}_D$, with $\mathcal{L}_D$ being defined in Lemma 2. Decompose $Y := \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$, where $Y_{11} \in \mathbf{R}^{(L+1)\times(L+1)}$ and $Y_{22} \in \mathbf{R}^{(n_P-1)\times(n_P-1)}$, according to*

$$C_2 = \begin{bmatrix} 0 & C_P \\ I_L & 0 \end{bmatrix} = \left[\begin{array}{cc:c} 0 & c_{P1} & 0 \\ I_L & 0 & 0 \end{array}\right] := \left[\begin{array}{c:c} C_{21} & 0 \end{array}\right], \tag{12}$$

and also decompose $Z := Y - X^{-1} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}$. Let

$$\bar{Y}_{11} := Y_{11} - Z_{11} + Z_{12}Z_{22}^+Z_{12}^T, \tag{13}$$

and construct

$$\bar{Y} := \begin{bmatrix} \bar{Y}_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}.$$

Then,

$$(X, \bar{Y}) \in \mathcal{L}_D; \tag{14}$$

$$\operatorname{rank}(\bar{Y} - X^{-1}) = \operatorname{rank} Z_{22} \leq n_P - 1 \tag{15}$$

hold, which imply that a feedback controller, $K_2(\lambda)$, with an order less than or equal to $n_P - 1$ can be constructed by applying the standard LMI algorithm to $(X, \bar{Y})$.

Similar to Theorem 1, we can obtain

COROLLARY 1 *The discrete-time $\mathcal{H}_\infty$ control problem for the generalized plant (7) with $C_F = I_{n_P}$ is solvable if and only if $\mathcal{L}_D \neq \emptyset$, where $\mathcal{L}_D$ is defined in Lemma 2, with $\mathcal{L}_C$ being simplified to*

$$\mathcal{L}_C := \{Y : Y = Y^T > 0, B_1^TYB_1 + D_{11}^TD_{11} - I < 0\}. \tag{16}$$

If it is solvable with the LMI solution $(X_0, Y_0) \in \mathcal{L}_D$, then $(X_0, X_0^{-1}) \in \mathcal{L}_D$ holds, from which it follows that there exists a static state feedback controller.

4. Design of Feedforward Controller

Let the coprime factorizations of the nominal plant $P_0(\lambda)$ and the local feedback controller $K_{2P}(\lambda)$ be

$$P_0 = \frac{N_P}{D_P}, \quad K_{2P} = \frac{N_{2K}}{D_{2K}}, \quad N_P, N_{2K}, D_P, D_{2K} \in \mathbf{R}[\lambda]. \tag{17}$$

Then, the transfer function of $P_0(\lambda)$ with local feedback $K_{2P}(\lambda)$ is $G(\lambda) = P_0(\lambda)/(1 + K_{2P}(\lambda)P_0(\lambda))$. Also, let its coprime factorization be

$$G = \frac{N_G}{D_G}, \quad N_G, D_G \in \mathbf{R}[\lambda]. \tag{18}$$

From Fig. 1, it is clear that

$$G_{yr} = N_G K_1, \tag{19}$$

$$G_{u_P r} = \frac{D_P N_G}{N_P} K_1. \tag{20}$$

The transfer function of the weighted Eq. (20) is

$$G_{u_W r} = W_u\left(\frac{D_P N_G}{N_P} K_1\right) := \frac{D_u}{N_u} K_1, \tag{21}$$

where $N_u, D_u \in \mathbf{R}[\lambda]$ are assumed to be coprime, and $W_u(\lambda)$ is a selected stable weighting transfer function.

Equations (19) and (21) show that the desired nominal input-output performance can be achieved by choosing a suitable feedforward controller, $K_1(\lambda)$.

In many designs, $K_1(\lambda)$ is chosen from $\mathbf{RH}_\infty$. However, in this paper, the assumption that information about future inputs can be used allows the class of $K_1(\lambda)$ to be expanded from $\mathbf{RH}_\infty$ to $\mathbf{R}_*$, the stable improper class. In what follows, a design method for a preview feedforward controller is described that introduces preview actions into $K_1(\lambda)$ to improve the tracking performance.

Unlike the method proposed by Funahashi and Katoh (1992), in this paper, $K_1(\lambda)$ is designed to satisfy the following conditions:

(1) Deadbeat condition: The tracking error $e[\lambda] = r[\lambda] - y[\lambda] = \sum_{i=-\infty}^{\infty} e_i \lambda^i$ is a finite polynomial in λ and z. i.e., $e \in \mathbf{R}[\lambda] \cup \mathbf{R}[z]$.

(2) Low-ripple condition: The transfer function (21) is a finite polynomial in λ and z. i.e., $G_{u_W r} \in \mathbf{R}[\lambda] \cup \mathbf{R}[z]$.

Decomposing $N_G(\lambda)$ and $D_u(\lambda)$ into

$$N_G = N_G^+ N_G^-; \quad D_u = D_u^+ D_u^-, \tag{22}$$

and letting $M \in \mathbf{R}[\lambda]$ be the greatest common divisor of $N_G^-(\lambda)$ and $D_u^-(\lambda)$ yields the following lemma.

LEMMA 3 *All feedforward controllers that satisfy the low-ripple condition are given by*

$$K_1 = \frac{N_u}{M}\bar{K}_1, \quad \bar{K}_1 \in \mathbf{R}[\lambda] \cup \mathbf{R}[z]. \tag{23}$$

Substituting Eq. (23) into Eqs. (19) and (21) yields

$$\begin{cases} G_{yr} = N\bar{K}_1; & N := \dfrac{N_G}{M} N_u \in \mathbf{R}[\lambda], \\ G_{u_W r} = D\bar{K}_1; & D := \dfrac{D_u}{M} \in \mathbf{R}[\lambda]. \end{cases} \tag{24}$$

Thus, the problem of designing the feedforward controller becomes that of designing $\bar{K}_1(\lambda)$.

Without loss of generality, assume

$$\begin{cases} D = a_0 + a_1\lambda + \cdots + a_n\lambda^n; & a_0, a_n \neq 0, \\ N = \lambda^m b(\lambda), & \\ b = b_0 + b_1\lambda + \cdots + b_l\lambda^l; & b_0, b_l \neq 0. \end{cases} \tag{25}$$

Now we are ready to construct a feedforward controller $\bar{K}_1^* \in \mathbf{R}[\lambda]$ with a minimum settling-time.

LEMMA 4 *The $\bar{K}_1$ in (23) that yields low-ripple deadbeat control with a minimum settling-time is given by*

$$\bar{K}_1^* = \frac{1 - \phi_R f^*}{N} \in \mathbf{R}[\lambda], \tag{26}$$

where f^* is the polynomial

$$f^* = f_0^* + f_1^*\lambda + \cdots + f_{m+l-1}^*\lambda^{m+l-1}, \tag{27}$$

and its coefficients are determined by the following algorithm. (For simplicity, we assume that $b(\lambda) = 0$ has only simple roots, which are denoted by $\lambda_1, \lambda_2, \cdots, \lambda_l$.) Algorithm:

Step 1 *$f_0^*, f_1^*, \cdots, f_{m-1}^*$ are determined by $\phi_R(\lambda)$ and the m-multiple original zero of $N(\lambda)$:*
If $L > m - 1$,

$$\begin{bmatrix} f_0^* \\ f_1^* \\ \vdots \\ f_{m-1}^* \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ \phi_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ \phi_{m-1} & \cdots & \phi_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \tag{28}$$

and if $L \leq m - 1$,

$$\begin{bmatrix} f_0^* \\ f_1^* \\ \vdots \\ f_L^* \\ \vdots \\ f_{m-1}^* \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 \\ \phi_1 & \ddots & & & & \\ \vdots & \ddots & \ddots & & & \\ \phi_L & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & & \phi_L & \cdots & \phi_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \tag{29}$$

Step 2 *$f_m^*, f_{m+1}^*, \cdots, f_{m+l-1}^*$ are determined by $\phi_R(\lambda)$ and the l-simple zeros, $\lambda_1, \lambda_2, \cdots, \lambda_l$, of $N(\lambda)$:*

$$\begin{bmatrix} f_m^* \\ f_{m+1}^* \\ \vdots \\ f_{m+l-1}^* \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{l-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{l-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_l & \cdots & \lambda_l^{l-1} \end{bmatrix}^{-1} \begin{bmatrix} g(\lambda_1) \\ g(\lambda_2) \\ \vdots \\ g(\lambda_l) \end{bmatrix}, \tag{30}$$

where

$$g(\lambda) := \frac{1}{\lambda^m \phi_R(\lambda)} - \sum_{i=0}^{m-1} f_i^* \lambda^{i-m}.$$

Based on the above results, the preview feedforward controller can be parameterized by the following theorem.

THEOREM 2 *All $\bar{K}_1^* \in \mathbf{R}[\lambda] \cup \mathbf{R}[z]$ that yield low-ripple deadbeat control are given by*

$$\bar{K}_1 = \bar{K}_1^* + \phi_R \tilde{K}_1, \quad \tilde{K}_1 \in \mathbf{R}[\lambda] \cup \mathbf{R}[z], \tag{31}$$

where $\bar{K}_1^*$ is obtained in Lemma 4, and $\tilde{K}_1$ is any polynomial in λ and z.

To design $\tilde{K}_1 \in \mathbf{R}[\lambda] \cup \mathbf{R}[z]$ that optimizes the transient response, first, choose two appropriate positive integers, p and q, and a non-zero polynomial

$$\begin{aligned} \tilde{K}_1 &= \tilde{k}_{-p}\lambda^p + \cdots + \tilde{k}_{-1}\lambda + \tilde{k}_0 + \tilde{k}_1 z + \cdots + \tilde{k}_q z^q \\ &= z^q(\tilde{k}_{-p}\lambda^{p+q} + \cdots + \tilde{k}_0\lambda^q + \cdots + \tilde{k}_q) := z^q \hat{K}_1, \end{aligned} \tag{32}$$

and let $n_{K1} := p + q$. Then,

$$\begin{aligned} e &= (1 - N\bar{K}_1)\frac{\bar{r}}{\phi_R} = f^*\bar{r} - z^{q-m} b\bar{r}\hat{K}_1 \\ &= \sum_{i=1}^{q-m} e_{-i} z^i + \sum_{j=0}^{L+l+m+p-1} e_j \lambda^j; \end{aligned} \tag{33}$$

$$u_W = G_{u_W r} r = D\bar{K}_1 \frac{\bar{r}}{\phi_R} = \frac{D\bar{r}\bar{K}_1^*}{\phi_R} + z^q D\bar{r}\hat{K}_1 \tag{34}$$

are obtained from Eqs. (2), (24) and (31). Decomposing the first term on the right side of the Eq. (34) yields

$$\begin{cases} \dfrac{D\bar{r}\bar{K}_1^*}{\phi_R} = \dfrac{\beta}{\phi_R} + \alpha \\ \beta = \beta_0 + \beta_1\lambda + \cdots + \beta_{L-1}\lambda^{L-1} \\ \alpha = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{L+n-2}\lambda^{L+n-2}. \end{cases} \tag{35}$$

So, the transient part of the weighted control input u_W is

$$\begin{aligned} \Delta u_W &= \alpha + z^q D\bar{r}\hat{K}_1 \\ &= \sum_{i=1}^{q} \Delta u_{W(-i)} z^i + \sum_{j=0}^{L+n+p-1} \Delta u_{Wj} \lambda^j. \end{aligned} \tag{36}$$

The performance index describing the transient response is defined to be

$$J := \sum_{i=-\infty}^{\infty} (|e_i|^2 + \rho^2 |\Delta u_{Wi}|^2), \tag{37}$$

and the parameters in $\tilde{K}_1$ of the feedforward controller are chosen to minimize J. The following transformation is introduced to simplify the description and calculation.

Let

$$\begin{aligned} &\bar{e} := \lambda^{q-m} e = \lambda^{q-m} f^*\bar{r} - b\bar{r}\hat{K}_1, \\ &\Delta\bar{u}_W := \lambda^q \Delta u_W = \lambda^q \alpha + D\bar{r}\hat{K}_1, \\ &L_\theta := L + l + n_{K1}; \quad L_\xi := L + n + n_{K1}\,. \end{aligned}$$

Then

$$\bar{e} = \sum_{i=0}^{L_\theta - 1} \bar{e}_i \lambda^i = \bar{e}^* - \theta\hat{K}_1 \in \mathbf{R}[\lambda], \tag{38}$$

and

$$\Delta\bar{u}_W = \sum_{i=0}^{L_\xi - 1} \Delta\bar{u}_{Wi} \lambda^i = \Delta\bar{u}_W^* + \xi\hat{K}_1 \in \mathbf{R}[\lambda], \tag{39}$$

where

$$\begin{cases} \bar{e}^* := \lambda^{q-m} f^*\bar{r} = \displaystyle\sum_{i=0}^{L_\theta - p - 2} \bar{e}_i^* \lambda^i \\ \Delta\bar{u}_W^* := \lambda^q \alpha = \displaystyle\sum_{i=0}^{L_\xi - p - 2} \Delta\bar{u}_{Wi}^* \lambda^i \\ \theta := b\bar{r} = \displaystyle\sum_{i=0}^{L+l-1} \theta_i \lambda^i \\ \xi := D\bar{r} = \displaystyle\sum_{i=0}^{L+n-1} \xi_i \lambda^i. \end{cases} \tag{40}$$

Hence, the performance index (37) becomes

$$J = \sum_{i=0}^{L_\theta - 1} |\bar{e}_i|^2 + \rho^2 \sum_{i=0}^{L_\xi - 1} |\Delta \bar{u}_{Wi}|^2. \qquad (41)$$

Then, according to $\mathcal{H}_2$ optimization method, the $\tilde{K}_1$ that minimizes J is given by the following theorem.

THEOREM 3 *The coefficient vector of $\tilde{K}_1 \in \mathbf{R}[\lambda] \cup \mathbf{R}[z]$ that minimizes the performance index J in (41) is*

$$\begin{bmatrix} \tilde{k}_q & \cdots & \tilde{k}_0 & \tilde{k}_{-1} & \cdots & \tilde{k}_{-p} \end{bmatrix}^T := F_1^{-1} F_2, \qquad (42)$$

where

$$F_1 = \begin{bmatrix} \Theta^T & -\rho \Xi^T \end{bmatrix} \begin{bmatrix} \Theta \\ -\rho \Xi \end{bmatrix}, \qquad (43)$$

$$F_2 = \begin{bmatrix} \Theta^T & -\rho \Xi^T \end{bmatrix} \begin{bmatrix} E^* \\ -\rho \Delta U_W^* \end{bmatrix}, \qquad (44)$$

$$\Theta = \begin{bmatrix} \theta_0 & & 0 \\ \vdots & \ddots & \\ \theta_{L+l-1} & \ddots & \theta_0 \\ & \ddots & \vdots \\ 0 & & \theta_{L+l-1} \end{bmatrix} \in \mathbf{R}^{L_\theta \times (n_{K1}+1)}, \qquad (45)$$

$$\Xi = \begin{bmatrix} \xi_0 & & 0 \\ \vdots & \ddots & \\ \xi_{L+n-1} & \ddots & \xi_0 \\ & \ddots & \vdots \\ 0 & & \xi_{L+n-1} \end{bmatrix} \in \mathbf{R}^{L_\xi \times (n_{K1}+1)}, \qquad (46)$$

$$E^* = \begin{bmatrix} \bar{e}_0^* & \bar{e}_1^* & \cdots & \bar{e}_{L_\theta - p - 2}^* & \mathbf{0}_{p+1} \end{bmatrix}^T, \qquad (47)$$

$$\Delta U_W^* = \begin{bmatrix} \Delta \bar{u}_{W0}^* & \Delta \bar{u}_{W1}^* & \cdots & \Delta \bar{u}_{W(L_\xi - p - 2)}^* & \mathbf{0}_{p+1} \end{bmatrix}^T \qquad (48)$$

with elements of the matrices and vectors of Eqs. (45) - (48) being given by Eq. (40). The minimum of J is

$$J_{\min} = \|E^*\|_2^2 + \rho^2 \|\Delta U_W^*\|_2^2 - F_2^T F_1^{-1} F_2. \qquad (49)$$

Regarding the relationship between J in Eq. (41) and $n_{K1} = p + q$, we have the following theorem.

THEOREM 4 *The performance index J in Eq. (41) is a monotonically decreasing function of p and q, which are parameters related to the settling steps and the preview steps, respectively. In addition,*

$$J_{opt}^{\infty} = \lim_{n_{K1} \to \infty} (\min J) = \left\| \begin{bmatrix} [T_{2i}^{\sim} T_1]_+ \\ H^{\sim} T_1 \end{bmatrix} \right\|_2^2, \qquad (50)$$

$$T_1 := \begin{bmatrix} \lambda^{q-m} f^* \bar{r} \\ \lambda^q \alpha \end{bmatrix}, \quad T_2 := \begin{bmatrix} b \bar{r} \\ -\rho D \bar{r} \end{bmatrix}, \qquad (51)$$

and $H \in \mathbf{RH}_\infty$ is chosen such that, for the inner-outer decomposition of T_2

$$T_2 = T_{2i} T_{2o}, \qquad (52)$$

$[T_{2i} \quad H]$ is a square matrix that satisfies

$$\begin{bmatrix} T_{2i}^{\sim} \\ H^{\sim} \end{bmatrix} [T_{2i} \quad H] = I. \qquad (53)$$

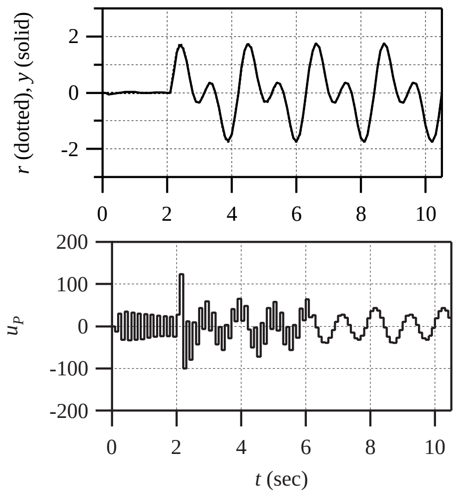

Figure 3: Optimal preview response for the nominal plant, $\zeta = 0.5$, ($p = 19, q = 21$, state feedback).

5. Simulations

Consider the plant described as

$$P(s) = \frac{\omega^2}{s^2 + 2\zeta s + \omega^2}; \quad 0 \le \zeta \le 1, \qquad (54)$$

where $\omega = 1$ rad/s. The nominal plant is $\zeta_0 = 0.5$, and

$$\begin{cases} A_P = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}; & B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_P = \begin{bmatrix} \omega^2 & 1 \end{bmatrix}, & \\ \Phi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; & \Gamma(t) = \dfrac{\zeta - \zeta_0}{\zeta_0}, \\ \Psi_A = \begin{bmatrix} 0 & -2\zeta_0\omega \end{bmatrix}; & \Psi_B = 0. \end{cases}$$

For the state and output feedback, we design TDF tracking controllers that robustly stabilize the control system for which the output tracks the periodic input

$$r(t) = \sin \frac{2\pi}{2.1} t + \sin \frac{4\pi}{2.1} t$$

without steady-state error at the sampling points. The periodicity of the reference input makes a repetitive control scheme suitable (Hara, *et al.*, 1988b; Tomizuka, *et al.*, 1989), and the internal model is

$$\phi_R(\lambda) = 1 - \lambda^L. \qquad (55)$$

The sampling period and the number of the steps of the repetitive controller are chosen to be $\tau = 0.1$ s and $L = 21$, respectively.

A feedback controller is designed under the conditions $Q_u^{1/2} = 0, Q_P^{1/2} = \left(C_P^T C_P\right)^{1/2} = diag\{\omega^2, 0\}, Q_R^{1/2} = I_{21}$; and a feedforward controller is designed under the conditions $W_u = D_G / D_P, \rho = 1$. In other words, the evaluated control input is chosen to be the input of the plant with the local feedback, $u[i]$, weighted by D_{2K} in (17).

Figures 3 - 5 show the simulation results for controllers designed according to the method proposed in this paper.

First, consider the state feedback case $C_F = I_2$. Corollary 1 yields a *static* state feedback controller K_2 According to Eq. (33), we choose $p = 19$, $q = 21$ to design a preview feedforward controller so as to make the outputs settle in the third period. Information about the inputs one period ahead are used in the design. It can be seen in Fig. 3 that, for the nominal plant, the tracking error is

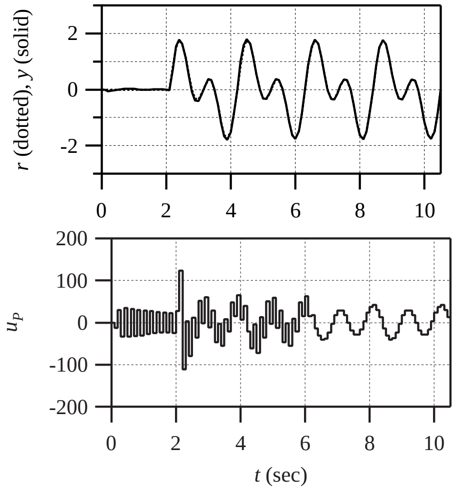

Figure 4: Optimal preview response for $\zeta = 0$ ($p = 19, q = 21$, state feedback).

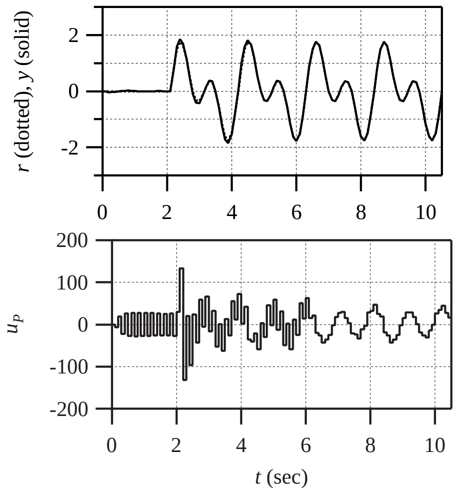

Figure 5: Optimal preview response for $\zeta = 0$ ($p = 19, q = 21$, output feedback).

very small when the reference is input, and the control input during the transient response is moderately restricted. When ζ ($0 \leq \zeta \leq 1$) is different from its nominal value of 0.5, the system still remains stable and its output tracks the reference input without steady-state error. As an example, Fig. 4 shows the simulation results for a plant with $\zeta = 0$.

Next, let us consider the output feedback case $C_F = C_P$. An output feedback controller is designed using Theorem 1. It has an order of *one* ($n_P - 1 = 1$). If we let $p = 19$, $q = 21$, we obtain a preview feedforward controller that settles the outputs in the third period.

As was seen in the state feedback case, the simulation results show that, the control system is stable for $0 \leq \zeta \leq 1$; and during the transient response, the tracking error is suppressed to a very low level, with the control input being moderately restricted. As an example, Fig. 5 shows the simulation results for $\zeta = 0$.

6. Conclusions

This paper describes a design method for digital tracking control systems for a continuous plant with structured uncertainties. A TDF tracking control system configuration is exploited. Regarding the design of the feedback controller, in order to robustly stabilize a plant with structured uncertainties, the design problem is first formulated as a sampled-data $\mathcal{H}_\infty$ control problem, and then transformed into an equivalent discrete-time $\mathcal{H}_\infty$ control problem. A static state feedback controller and a reduced-order output feedback controller with an order no greater than that of the plant minus one have been designed by using the LMI-based $\mathcal{H}_\infty$ control approach. Regarding the design of the feedforward controller, parameterization of the feedforward controller is carried out based on the previously designed feedback controller, in which the free parameter is chosen to achieve the desired transient response using a preview strategy. The validity of the method has been demonstrated by simulations.

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