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Repetitive Control System with Variable Structure Controller

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This paper presents a new design method for repetitive control that employs discrete-time variable structure (VS) control algorithm. First, an extended system containing an internal model of repetitive control is derived. Based on that, a sliding mode is designed, and a sliding sector is defined around it. A variable structure control law is then designed so that the state converges inside the sliding sector in a finite numbers of steps and then stays in the sector forever. It ensures that there exists a Lyapunov function that decreases monotonically in the state space. Thus the system is asymptotically stable. Simulation results show that the proposed control system provides good performance.

1 Introduction

Many control systems perform cyclic operations. For example, in noncircular cutting with a lathe, the position of the cutting tool has to track a given periodic signal. Repetitive control is a very useful strategy for tracking periodic reference inputs (Hara, *et al.*, 1988; Tomizuka, *et al.*, 1989). Various control algorithms can be used to design a repetitive controller, such as LQR, $\mathcal{H}_\infty$, and so on. Among them, variable structure (VS) control (Utkin, 1992) is quite effective, and discrete-time VS control systems with a boundary layer, or sliding sector, have been designed for discrete-time systems (Drakunov and Utkin, 1989; Yu and Potts, 1991; Bartolini, *et al.*, 1992; Pieper and Surgenor, 1992; Young, et al., 1996; Furuta and Pan, 2000; Pan, *et al.*, 2000).

This paper proposes a new design method for repetitive control using a discrete-time VS control system with an invariant sliding sector. First, an extended system containing an internal model of repetitive control is derived.

Based on that, a sliding mode is designed, and a sliding sector is defined around it. A VS control law is then designed in such a way that the state converges inside the sliding sector in a finite number of steps and then stays there forever, i.e., the sliding sector is invariant. This control law ensures that there exists a Lyapunov function that decreases monotonically inside and outside the sliding sector, i.e., in the state space. Thus, the system is asymptotically stable.

The text is arranged as follows: Section 2 formulates the problem; Section 3 defines the invariant sliding sector and explains the design of the corresponding VS control law; Section 4 gives some simulation results.

2 Problem Formulation

The configuration of the proposed repetitive control system is shown in Fig. 1. Consider the following discrete-time plant:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k), \end{cases} \tag{1}$$

where $x(k) \in \mathbf{R}^n$, $y(k) \in \mathbf{R}$, and $u(k) \in \mathbf{R}$ are the state, output, and control input, respectively. The goal is to design a repetitive controller that provides good tracking performance for a periodic reference. To make this design problem solvable, three assumptions are made.

Assumtion 1 *(A, B) is controllable and (C, A) is observable.*

Assumtion 2 *The plant (A, B, C) has no zeros in common with $1 - z^{-L}$, where L is the number of steps of the repetitive controller.*

Assumtion 3 *The state $x(k)$ is available.*

We use the method proposed by Egami and Tsuchiya (1992) to construct an extended plant containing an internal model of repetitive control.

For a periodic input $r(k)$, the tracking error is given by

$$e(k) = r(k) - y(k). \tag{2}$$

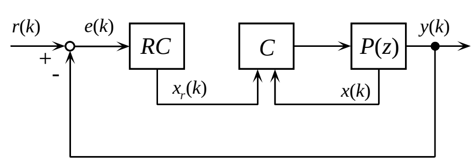

Figure 1: Configuration of repetitive control system.

Multiplying both sides of (2) by $(1 - z^{-L})$ and shifting the equation one step forward yields

$$(1 - z^{-L})e(k+1) = (1 - z^{-L})r(k+1) - (1 - z^{-L})y(k+1),$$

i.e.,

$$\begin{aligned} e(k+1) = z^{-L}e(k+1) + (1 - z^{-L})r(k+1) - CA\{(1 - z^{-L})x(k)\} \\ -CB\{(1 - z^{-L})u(k)\}. \end{aligned} \tag{3}$$

On the other hand, multiplying both sides of the state equation of (1) by $(1 - z^{-L})$ gives

$$(1 - z^{-L})x(k+1) = A\{(1 - z^{-L})x(k)\} + B\{(1 - z^{-L})u(k)\}. \tag{4}$$

Defining

$$\begin{cases} x_r(k) := [\, e(k) \quad e(k-1) \quad \cdots \quad e(k-L+1)\,]^T \\ \Delta x(k) := (1 - z^{-L})x(k) \\ x_e(k) := [\, x_r^T(k) \quad \Delta x^T(k)\,]^T \\ u_e(k) := (1 - z^{-L})u(k) \end{cases}$$

and combining (3) and (4), we have

$$x_e(k+1) = \Phi x_e(k) + \Gamma u_e(k) + \Gamma_r \left\{(1 - z^{-L})r(k+1)\right\}, \tag{5}$$

$$\Phi = \begin{bmatrix} 0 & \cdots & 0 & 1 & -CA \\ 1 & 0 & \cdots & 0 & \mathbf{0} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & 1 & 0 & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & A \end{bmatrix}, \Gamma = \begin{bmatrix} -CB \\ 0 \\ \vdots \\ 0 \\ B \end{bmatrix}, \Gamma_r = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Note that the period of the repetitive controller, L, is the same as that of the reference input, $(1 - z^{-L})r(k+1) = 0$ for all $k \geq L-1$. So, system (5) is affected by the input $(1 - z^{-L})r(k+1)$ only in the first period, which means that the effect of this input can be handled in the initial state. For this reason, the term $\Gamma_r(1 - z^{-L})r(k+1)$ in (5) is omitted, and (5) is rewritten as

$$x_e(k+1) = \Phi x_e(k) + \Gamma u_e(k). \tag{6}$$

which is an extended plant containing an internal model of repetitive control. Now, the design problem becomes: *Design a controller that stabilizes the plant (6).*

3 Design of VS Controller

According to Assumptions 1 and 2, the plant (6) is controllable. Thus, there exists a nonsingular matrix $T \in \mathbf{R}^{(n+L)\times(n+L)}$ that converts the plant into the following controllable canonical form:

$$\bar{x}_e(k+1) = \bar{\Phi}\bar{x}_e(k) + \bar{\Gamma}u_e(k), \tag{7}$$

i.e.,

$$\left[\begin{array}{c}\bar{x}_{e1}(k+1)\\ \hline \bar{x}_{e2}(k+1)\end{array}\right] = \left[\begin{array}{c|c}\bar{\Phi}_{11} & \bar{\Phi}_{12}\\ \hline \bar{\Phi}_{21} & \bar{\Phi}_{22}\end{array}\right]\left[\begin{array}{c}\bar{x}_{e1}(k)\\ \hline \bar{x}_{e2}(k)\end{array}\right] + \left[\begin{array}{c}\bar{\Gamma}_1\\ \hline \bar{\Gamma}_2\end{array}\right]u_e(k),$$

where

$$\bar{\Phi} = T^{-1}\Phi T = \left[\begin{array}{cccc|c} 0 & 1 & & & 0\\ \vdots & & \ddots & & \vdots\\ 0 & & & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\\ \hline -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n+L-2} & -\alpha_{n+L-1}\end{array}\right], \; \bar{\Gamma} = T^{-1}\Gamma = \left[\begin{array}{c}0\\ \vdots\\ 0\\ 0\\ \hline 1\end{array}\right].$$

The stabilizing controller is designed based on (7). First, a hyperplane, i.e. a sliding mode, is designed using the LQ optimal control method. Then a sliding sector is defined. Finally, an additional control law, i.e., a VS control law is proposed to drive the state of the system into the sliding sector in a finite time and to ensure that the sliding sector is invariant with a Lyapunov function decreasing in the state space.

3.1 Sliding mode

The sliding mode, i.e., hyperplane, is defined to be

$$\sigma(k) = \hat{S}\bar{x}_e(k) = \bar{S}\bar{x}_{e1}(k) + \bar{x}_{e2}(k) = 0, \tag{8}$$

where $\hat{S} = [\,\bar{S} \quad 1\,] \in \mathbf{R}^{n+L}$ is chosen to ensure that the state $\bar{x}_e(k) = [\,\bar{x}_{e1}^T(k) \quad \bar{x}_{e2}^T(k)\,]^T$ is asymptotically stable in the sliding mode.

When the state is in this mode, the following holds:

$$\sigma(k+1) = \sigma(k) = 0. \tag{9}$$

Therefore the equivalent control law, which guarantees the above equation, is given by

$$u_{eq}(k) = -(\hat{S}\bar{\Gamma})^{-1}(\hat{S}\bar{\Phi}\bar{x}_e(k) - \sigma(k)) = -(\hat{S}\bar{\Phi}\bar{x}_e(k) - \sigma(k)), \tag{10}$$

which can be rewritten as:

$$u_{eq}(k) = -(\bar{S} - \bar{S}\bar{\Phi}_{11} - \bar{\Phi}_{21})\bar{x}_{e1}(k) + (1 - \bar{S}\bar{\Phi}_{12} - \bar{\Phi}_{22})\bar{x}_{e2}(k). \qquad (11)$$

¿From the equation $\sigma(k) = 0$ in the sliding mode,

$$\bar{x}_{e2}(k) = -\bar{S}\bar{x}_{e1}(k) \qquad (12)$$

is obtained. Substituting (11) and (12) into (7) yields a reduced-order system in the sliding mode:

$$\bar{x}_{e1}(k+1) = \bar{\Phi}_{11}\bar{x}_{e1}(k) + \bar{\Phi}_{12}\bar{x}_{e2}(k) = (\bar{\Phi}_{11} - \bar{\Phi}_{12}\bar{S})\bar{x}_{e1}(k), \qquad (13)$$

which is called a sliding-mode system (Yasuda, *et al.*, 1998).

Clearly, (13) is asymptotically stable iff all eigenvalues of $(\bar{\Phi}_{11} - \bar{\Phi}_{12}\bar{S})$ are inside the open unit circle in the complex z plane. The parameter $\bar{S}$ can be designed by various methods, e.g., eigenvalue assignment, optimal control, and $\mathcal{H}_\infty$ control. In this paper, the LQ optimal control method is employed. A discrete-time quadratic performance index is defined as

$$J = \sum_{k=0}^{\infty}\{\bar{x}_{e1}^T(k)\bar{Q}\bar{x}_{e1}(k) + \bar{x}_{e2}^T(k)\bar{x}_{e2}(k)\}, \qquad (14)$$

where $\bar{Q} \in \mathbf{R}^{(n+L-1)\times(n+L-1)}$ is a symmetric positive definite matrix.

It is clear that the pair $(\bar{\Phi}_{11}, \ \bar{\Phi}_{12})$ is controllable. Thus an optimal solution to minimize (14) for the plant (13) exists and has the form

$$\bar{x}_{e2}(k) = -F\bar{x}_{e1}(k).$$

The gain matrix F is defined as

$$F = (1 + \bar{\Phi}_{12}^T\bar{P}\bar{\Phi}_{12})^{-1}\bar{\Phi}_{12}^T\bar{P}\bar{\Phi}_{11}, \qquad (15)$$

where $\bar{P}$ is a symmetric positive definite matrix that satisfies the following discrete-time Riccati equation:

$$\bar{P} = \bar{Q} + \bar{\Phi}_{11}^T\bar{P}\bar{\Phi}_{11} - \bar{\Phi}_{11}^T\bar{P}\bar{\Phi}_{12}(1 + \bar{\Phi}_{12}^T\bar{P}\bar{\Phi}_{12})^{-1}\bar{\Phi}_{12}^T\bar{P}\bar{\Phi}_{11}. \qquad (16)$$

If $\bar{S}$ is chosen to be

$$\bar{S} = F, \qquad (17)$$

then the sliding mode system (13) is asymptotically stable.

Furthermore, the sliding mode represented by the state $x_e(k)$ is given by

$$\sigma(k) = Sx_e(k), \quad S := [\,\bar{S} \quad 1\,]\,T^{-1}. \qquad (18)$$

3.2 Sliding sector

With a sliding mode, a continuous-time VS controller is designed such that, once the state converges to the sliding mode, it then stay there forever. To guarantee that, the switching frequency must be infinity. However for a discrete-time VS control system, this is not realizable. In general, discrete-time sliding-mode controllers (e.g., Young, *et al.*, 1996; Furuta, 1990) are designed so that the state moves into a boundary layer, or sliding sector, around the sliding mode in a finite time rather than into the sliding mode itself.

In this paper, a sliding sector around the sliding mode $\sigma(k) = 0$ is defined to be

$$\Psi = \{x_e(k) | \sigma^2(k) < \delta^2(k)\}, \tag{19}$$

where $\delta^2(k)$ is a quadratic function of $x_e(k)$ and is defined to be

$$\delta^2(k) = \gamma x_e^T(k) \Pi x_e(k). \tag{20}$$

γ is a positive constant and $\Pi \in \mathbf{R}^{(n+L)\times(n+L)}$ is a symmetric positive definite matrix, which will be determined in the next subsection in such a way that the sliding sector (19) is invariant with some VS control law.

3.3 VS control law

Give the sliding sector (19), a discrete-time VS control law should be designed such that:

1. the state moves into the sliding sector in a finite time;
2. stay in the sliding sector after entry, i.e., the sliding sector is invariant;
3. some Lyapunov function decreases monotonically in the state space.

To guarantee the above requirements, we use the VS control law designed to be

$$u_e(k) = \begin{cases} -(S\Phi x_e(k) - \alpha\sigma(k)), & x_e(k) \notin \Psi, \\ -(S\Phi x_e(k) - \beta\sigma(k)), & x_e(k) \in \Psi, \end{cases} \tag{21}$$

where α and β are constants and satisfy

$$0 \leq |\alpha| \leq |\beta| < 1.$$

Outside the sliding sector, it follows from (21) that

$$\sigma(k+1) = \alpha\sigma(k).$$

Based on the above equation, it can be shown that, if α is chosen close enough to zero, then the state will move into the sliding sector (19) in a finite number of steps even though it takes an infinite number of steps to converge to the sliding mode (8). And if α is chosen to be zero, then the state will move into the sliding sector in only one step.

Inside the sliding sector, the control law (21) ensures that the closed-loop system of the plant (6) is stable with one eigenvalue being β and the others being determined by the LQ optimal feedback control law (16) for the sliding mode system (13). So, for any symmetric positive definite matrix $\Theta \in \mathbf{R}^{(n+L)\times(n+L)}$, there exist a symmetric positive definite matrix $\Pi \in \mathbf{R}^{(n+L)\times(n+L)}$ and two positive constants γ_1 and γ_2 $(0 < \gamma_2 < \gamma_1 < 1)$ such that

$$-\gamma_1\Pi < \tilde{\Phi}^T\Pi\tilde{\Phi} - \Pi = -\Theta < -\gamma_2\Pi, \tag{22}$$

where $\tilde{\Phi} \in \mathbf{R}^{(n+L)\times(n+L)}$ is the system matrix of the closed-loop system:

$$\tilde{\Phi} = \Phi - \Gamma(S\Phi - \beta S).$$

It follows from (22) that

$$(1-\gamma_1)\delta^2(k) < \delta^2(k+1) < (1-\gamma_2)\delta^2(k).$$

Inside the sliding sector, i.e., $\sigma^2(k) \le \delta^2(k)$, the following holds:

$$\sigma^2(k+1) - \delta^2(k+1) < -(1-\beta^2-\gamma_1)\delta^2(k).$$

Therefore, if we choose β to satisfy $\beta^2 + \gamma_1 < 1$, then

$$\sigma^2(k+1) - \delta^2(k+1) < 0,$$

which means that the state will stay inside the sliding sector (19) forever once it moves there.

To check the stability of the proposed VS control system, we first rewrite the VS control law (21) as

$$u_e(k) = -(S\Phi x_e(k) - \eta\sigma(k)), \tag{23}$$

where

$$\eta = \begin{cases} \alpha, & x_e(k) \notin \Psi, \\ \beta, & x_e(k) \in \Psi. \end{cases} \tag{24}$$

Now, if we define a state variable $z(k) \in \mathbf{R}^{n+L}$ to be

$$z(k) = \left[\, \bar{x}_{e1}^T(k)\ \sigma(k) \,\right]^T,$$

then the closed-loop system with the VS control law (21) is

$$z(k+1) = \begin{bmatrix} \bar{\Phi}_{11} - \bar{\Phi}_{12}\bar{S} & \bar{\Phi}_{12} \\ 0 & \eta \end{bmatrix} z(k),$$

where $\bar{\Phi}_{11}$, $\bar{\Phi}_{12}$, and $\bar{S}$ are defined in (7) and (8).

Finally, we choose the Lyapunov function

$$L(k) = z^T(k) \begin{bmatrix} \bar{P} & 0 \\ 0 & \bar{\gamma} \end{bmatrix} z(k), \tag{25}$$

where $\bar{P}$ is the symmetric positive definite solution of (16) and $\bar{\gamma}$ is a positive constant. Thus, the following holds:

$$\begin{aligned} \Delta L(k) = L(k+1) - L(k) &= -z^T(k) \begin{bmatrix} \bar{Q} + \bar{S}^T \bar{S} & -\bar{S}^T \\ -\bar{S} & -\bar{\Phi}_{12}^T \bar{P} \bar{\Phi}_{12} + \bar{\gamma}(1-\eta^2) \end{bmatrix} z(k) \\ &:= -z^T(k) \bar{\Theta} z(k). \end{aligned}$$

As it is assumed that $0 \leq |\alpha| \leq |\beta| < 1$, we have

$$1 - \eta^2 > 0.$$

Since

$$\bar{Q} + \bar{S}^T \bar{S} > 0,$$

from Schur complement, if we choose the positive constant $\bar{\gamma}$ so that

$$\bar{\gamma} > \frac{\bar{\Phi}_{12}^T \bar{P} \bar{\Phi}_{12} + \bar{S}[\bar{Q} + \bar{S}^T \bar{S}]^{-1} \bar{S}^T}{1 - \eta^2} > 0, \tag{26}$$

then the matrix $\bar{\Theta}$ is positive definite, i.e., the following inequality holds

$$\Delta L(k) \leq 0, \quad \forall x_e(k) \in \mathbf{R}^{n+L}. \quad (\Delta L(k) = 0 \text{ only if } x_e(k) = 0)$$

which means that the proposed VS control system is asymptotically stable.

Outside the sliding sector, the symmetric matrix $\Theta \in \mathbf{R}^{(n+L)\times(n+L)}$ is positive semi-definite if α is chosen to be zero. When α is larger than zero, Θ is positive definite if the positive constant η is large enough. So,

$$\Delta L(k) \geq 0, \quad \forall x_e(k) \in \mathbf{R}^{n+L}.$$

Note that $\Delta L(k) = 0$ only when the state is at the origin and/or $\alpha = 0$. When $\alpha = 0$, the state moves into the sliding sector in only one step, so $\Delta L(k) = 0$ only for one step. Thus the proposed VS control system is asymptotically stable.

Summarizing the above results, we obtain the following theorem.

Theorem 1 *The VS control law in equation (21) ensures that the state moves into the sliding sector (19) in a finite number of steps, the sliding sector (19) is invariant, and there exists a Lyapunov function that decreases monotonically in the state space. Therefore, the proposed VS control system for repetitive control is asymptotically stable.*

4 Numerical Example

Consider the following second-order plant:

$$\begin{cases} P(s) = \frac{\omega^2}{s^2+2\zeta\omega s+\omega^2}, \\ \omega = 1\,\mathrm{rad/s}; \quad \zeta = 0.5. \end{cases} \tag{27}$$

We assumed

Sampling inerval :	$\tau = 0.1\,\mathrm{s}$,
Number of steps of repetitive controller :	$L = 21$,
Periodic reference input :	$r(k) = \sin\frac{2\pi}{21}k + \sin\frac{4\pi}{21}k$;

let

$$\bar{Q} = I_{22\times 22}, \quad \gamma = 0.1414, \quad \alpha = 0.1, \quad \beta = 0.3; \tag{28}$$

and designed a repetitive control system using the approach developed in Sections 3.

Fig. 2 shows some simulation results for the nominal plant. The output reaches the steady state in 21 steps. This system is also robust even if parameter uncertainties exist. For example, Fig. 3 shows some simulation results when the system matrix has the following uncertainties:

$$\Delta A = B \times [\,0.1 \quad \cdots \quad 0.1\,], \quad (A = A_0 + \Delta A). \tag{29}$$

5 Conclusion

This paper presents a design method for repetitive control using the discrete-time VS control algorithm. First, an extended system containing an internal model of repetitive control is derived. Based on that, a repetitive control design method with a discrete-time VS controller is proposed. A sliding mode is designed using the discrete-time LQ optimal control. A sliding sector is defined around the sliding mode, and a VS control law is derived that drives the state of the system into the sliding sector and ensures the invariance of that sector. Simulation results show the proposed algorithm to be very effective.

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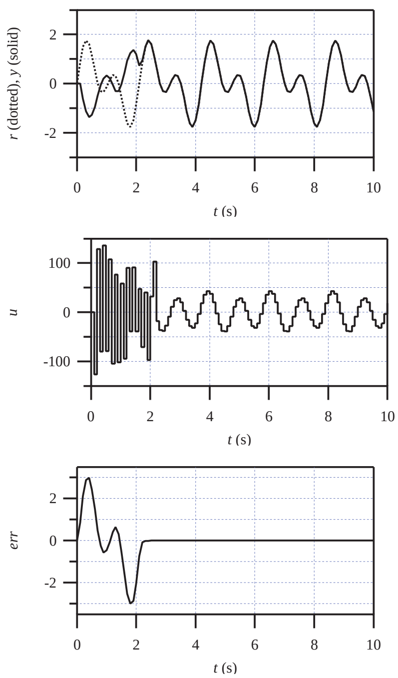

Figure 2: Simulation results for the nominal plant.

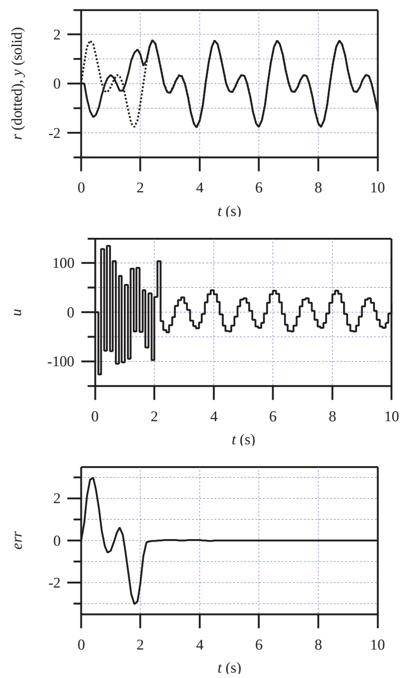

Figure 3: Simulation results for a plant containing uncertainties.

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